Module Homomorphisms

A mapping $f : M \to N$ between $A$-modules $M$, $N$ is called an

$A$-module homomorphism

or

$A$-linear

if it respects addition and scalar multiplication, that is, for $x, y \in M$ and $a \in A$

$$f(x + y) = f(x) + f(y) \quad , \quad f(ax) = af(x)$$
(so \( f \) is an abelian group homomorphism which respects the action of the ring).

If \( A \) is a field then an \( A \)-module homomorphism is just a linear transformation.

Put

\[
\text{Hom}_A(M, N) = \{ \text{\( A \)-module homomorphisms : } M \rightarrow N \}\]

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also written $\text{Hom}(M, N)$ if $A$ is clear from context.

Define **pointwise** addition and scalar multiplication on $\text{Hom}(M, N)$:

$$\forall f, g \in \text{Hom}(M, N) \quad \forall a \in A$$

$$(f + g)(x) = f(x) + g(x) \quad , \quad (af)(x) = af(x).$$

Routine to check:

under these operations $\text{Hom}_A(M, N)$ becomes an $A$-module.
Induced homomorphisms:

Suppose $M', M, N, N'$ are $A$-modules and $u : M' \to M$, $v : N \to N'$ are $A$-module homomorphisms.

Composition of mappings induces homomorphisms between appropriate Hom modules.
Define

\[ \overline{u} : \text{Hom} \left( M, N \right) \rightarrow \text{Hom} \left( M', N \right) \]

by

\[ \overline{u}(f) = f \circ u . \]
Define

$$\overline{v} : \text{Hom} \,(M, N) \rightarrow \text{Hom} \,(M, N')$$

by

$$\overline{v}(f) = v \circ f.$$
Easy to check:

\[ \overline{u}, \overline{v} \] are themselves \( A \)-module homomorphisms

and we say that \( \overline{u}, \overline{v} \) are \textbf{induced} from \( u, v \) respectively.

**Example:** Suppose that \( A \) is a field and \( M', M, N, N' \) are finite dimensional vector spaces of dimension \( m', m, n, n' \) respectively.
Linear transformations may be identified with matrices

So

\[ \text{Hom} \left( M, N \right) \equiv \text{Mat} \left( n, m \right) \]
\[ = \{ \text{n \times m matrices over } A \} , \]

\[ \text{Hom} \left( M', N \right) \equiv \text{Mat} \left( n, m' \right) , \]

\[ \text{Hom} \left( M, N' \right) \equiv \text{Mat} \left( n', m \right) . \]
Let $u : M' \rightarrow M$, $v : N \rightarrow N'$ be linear transformations. Regard

$u$ as an $m \times m'$ matrix,

and

$v$ as an $n' \times n$ matrix.

Then the induced homomorphism become simply pre and post-multiplication respectively by matrices:
$\overline{u} : \text{Mat} (n, m) \to \text{Mat} (n, m')$

$x \mapsto x \cdot u$;

$\overline{v} : \text{Mat} (n, m) \to \text{Mat} (n', m)$

$x \mapsto v \cdot x$. 
**Observation:** For any $A$-module $M$, 

$$\text{Hom}_A(A, M) \cong M.$$ 

**Proof:** It is routine to check that 

$$f \mapsto f(1) \quad \forall f \in \text{Hom}_A(A, M)$$ 

is a bijective $A$-module homomorphism.