(⇐) Suppose conversely, for all $A$-modules $N$, that

\[
0 \longrightarrow \text{Hom} (M'', N) \longrightarrow \text{Hom} (M, N) \longrightarrow \text{Hom} (M', N)
\]

is exact.

(i) We show $\nu$ is surjective:

Put $N = M''/\text{im} \nu$ and let $f : M'' \rightarrow N$ be the natural map.

Observe that $\nu(f) = f \circ \nu = 0$, the zero map, by definition of $f$,
so \( f = 0 \), since \( \overline{v} \) is injective.

But this means \( M'' = \text{im} \, v \), that is, \( v \) is surjective.

(ii) We show \( \text{im} \, u \subseteq \text{ker} \, v \):

Put \( N = M'' \) and let \( f : M'' \to N \) be the identity mapping. Then

\[
0 = (\overline{u} \circ \overline{v})(f) = f \circ v \circ u = v \circ u
\]

(since \( \text{im} \, \overline{v} = \ker \, \overline{u} \)), which proves \( \text{im} \, u \subseteq \ker \, v \).
(iii) We show \( \ker v \subseteq \text{im} \, u \):

Put \( N = M/\text{im} \, u \) and let \( f : M \to N \) be the natural map.

Certainly \( \overline{u}(f) = f \circ u = 0 \) (by definition of \( f \)).

so \( f \in \ker \overline{u} = \text{im} \, \overline{v} \), yielding

\[
f = \overline{v}(g) = g \circ v
\]

for some \( g \in \text{Hom} \, (M'', N) \).
But \( \ker(g \circ v) \supseteq \ker v \), so

\[
\text{im } u = \ker f = \ker(g \circ v) \supseteq \ker v .
\]

Facts (i), (ii), (iii) establish that

\[
\begin{array}{ccc}
\text{u} & \text{v} \\
M' & \rightarrow & M & \rightarrow & M'' & \rightarrow & 0
\end{array}
\]

is exact, and (1) of the Theorem is proved.
Let

\[
\begin{array}{cccccc}
0 & \rightarrow & M' & \rightarrow & M & \rightarrow & M'' & \rightarrow & 0 \\
& & f' & \downarrow & f & \downarrow & f'' & \\
0 & \rightarrow & N' & \rightarrow & N & \rightarrow & N'' & \rightarrow & 0 \\
& & u' & & u' & & v' & \\
\end{array}
\]

be a commutative diagram of $A$-modules and homomorphisms, with exact rows.
Theorem: With the above, there exists an exact sequence

\[
0 \rightarrow \ker f' \rightarrow \ker f \rightarrow \ker f'' \\
\overline{u} \overline{v} \quad d \quad \overline{u}' \overline{v}'
\]

\[
coker f' \rightarrow coker f \rightarrow coker f'' \rightarrow 0
\]

for some homomorphism \( d \).
In the above diagram

\( \overline{u} , \overline{v} \) denote the restrictions of \( u , v \) respectively, and

\( \overline{u}' , \overline{v}' \) are induced by composites of \( u' , v' \) respectively with natural maps.

**Proof:** Define

\[
d : \ker f'' \rightarrow \coker f' = N/\text{im } f'
\]

as follows:
Let \( x'' \in \ker f'' \). Then, since \( v \) is onto,

\[
x'' = v(x) \quad \exists x \in M
\]

so

\[
v'(f(x)) = f''(v(x)) = f''(x'') = 0,
\]

yielding

\[
f(x) \in \ker v' = \text{im} \ u',
\]
whence

\[ f(x) = u'(y') \quad \exists y' \in N'. \]

Now put

\[
\begin{align*}
    d(x'') &= y' + f'(M') \in N'/\text{im } f'.
\end{align*}
\]

(i) Check that \( d \) is well-defined:
This is a simple **exercise**, using exactness at $M$, commutativity of the first square and the fact that $u'$ is injective.

(ii) Check that $d$ is a module homomorphism:

This follows easily, tracing through the definition of $d$ and using the fact that each of $v$, $f$ and $u'$ are homomorphisms.
(iii) Check exactness at \( \ker f' \) and \( \coker f'' \):

This is immediate because \( \overline{u} \) is injective (restriction of an injective map) and \( \overline{v}' \) is surjective (induced by a surjective map).

(iv) Check exactness at \( \ker f \):

If \( x \in \ker \overline{v} \) then \( x \in \ker v = \text{im} v \), so

\[
x = u(x') \quad \exists x' \in M'
\]
and

\[ u'(f'(x')) = f(u(x')) = f(x) = 0, \]

so

\[ f'(x') = 0 \quad \text{(since } u' \text{ is injective)} \]

yielding \( x' \in \ker f' \), whence

\[ x = u(x') = \overline{u}(x') \in \text{im } \overline{u}. \]

Thus \( \ker \overline{u} \subseteq \text{im } \overline{u} \).
Conversely, if \( x \in \text{im } \overline{u} \) then

\[
x = \overline{u}(x') = u(x') \quad \exists x' \in \ker f'
\]

so

\[
f(x) = f(u(x')) = u'(f'(x')) = u'(0) = 0 ,
\]

so

\[
x \in \ker f \cap \text{im } u = \ker f \cap \ker v
\]

so \( x \in \ker \overline{v} \). Thus \( \text{im } \overline{u} = \ker \overline{v} \), and equality holds.
Check exactness at $\text{coker } f$:

This is left as an exercise.

Check exactness at $\ker f''$:

Suppose $x'' \in \ker d$, so

$$x'' = v(x) \quad \exists x \in M, \quad f(x) = u'(y') \quad \exists y' \in N'$$
and

\[ f'(M') = d(x'') = y' + f'(M') . \]

Thus \( y' \in f'(M') \), so

\[ y' = f'(x') \quad \exists x' \in M' \]

yielding

\[ f(x) = u'(y') = u'(f'(x')) = f(u(x')) , \]
so \( x - u(x') \in \ker f \). Observe now that

\[
\overline{v}(x-u(x')) = v(x) - v(u(x'')) = v(x) = x'',
\]

proving \( \ker d \subseteq \im \overline{v} \).

Conversely, if \( x'' \in \im \overline{v} \) then

\[
x'' = \overline{v}(x) = v(x) \quad \exists \ x \in \ker f
\]
so \( f(x) = 0 = u'(0) \), so (by definition)

\[
d(x'') = 0 + f'(M') = f'(M'),
\]

proving \( x'' \in \ker d \), whence \( \text{im } \bar{v} = \ker d \).

(vii) **Check exactness at** \( \text{coker } f' \):

This is left as an **exercise**.

The Theorem is proved.
Exercise: In the earlier diagram with commuting squares and exact rows, find an example in which each of

\[ \ker f', \quad \ker f, \quad \ker f'', \quad \text{coker } f', \quad \text{coker } f, \quad \text{coker } f'' \]

is not a zero module, and each of

\[ \overline{u}, \quad \overline{v}, \quad d, \quad \overline{u}', \quad \overline{v}' \]

is not a zero homomorphism.
Let $\mathcal{C}$ be a class of $A$-modules containing the zero module.

Call $\lambda : \mathcal{C} \to \mathbb{Z}$ additive if, for each short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

where $M', M, M'' \in \mathcal{C}$ we have

$$\lambda(M) = \lambda(M') + \lambda(M'').$$
Note that

\[ 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \]

is exact, so \( \lambda(0) = \lambda(0) + \lambda(0) \), yielding

\[ \lambda(0) = 0. \]

**Example:** Let \( A = F \) be a field and \( C \) the class of all finite dimensional vector spaces over \( F \).
If
\[ 0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \]
is exact, then
\[ M'' \cong M / \ker g = M / f(M') \]
so (by the Rank-Nullity Theorem)
\[ \dim M'' = \dim(M / f(M')) = \dim M - \dim f(M') \]
\[ = \dim M - \dim M' , \]
which proves \( \dim : C \rightarrow \mathbb{Z} \) is additive.
Example: Let $\mathcal{C}$ denote the class of all finite abelian groups, regarded as $\mathbb{Z}$-modules.

Let $\mathcal{P}$ be some given set of primes (possibly all primes). If $A \in \mathcal{C}$ then

$$|A| = \left( \prod_{p \in \mathcal{P}} p^{\alpha_p} \right) q$$

where $q$ is coprime to all elements of $\mathcal{P}$.

Define $\lambda(A) = \sum_{p \in \mathcal{P}} \alpha_p$.

Clearly $\lambda$ is additive.
**Example:** Let $C$ denote the class of all finitely generated abelian groups, regarded as $\mathbb{Z}$-modules.

If $A \in C$ then $A \cong \mathbb{Z}^n \oplus B$ for some $n \geq 0$ and finite abelian group $B$.

Define $\lambda(A) = n = \text{torsion free rank}$.

**Exercise:** Prove $\lambda$ is additive.
Proposition: Let

\[
0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \ldots \longrightarrow M_n \longrightarrow 0
\]

be exact where all modules and kernels belong to \( \mathcal{C} \), and let \( \lambda \) be additive. Then

\[
\sum_{i=0}^{n} (-1)^i \lambda(M_i) = 0 .
\]
Proof: We have that

\[
\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\
& \lrcorner & \lrcorner & \lrcorner & \lrcorner & \lrcorner & & \\
0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & \cdots & \longrightarrow & N_n \\
& \lrcorner & \lrcorner & \lrcorner & \lrcorner & \lrcorner & & \\
0 & \longrightarrow & M_0 & \longrightarrow & M_1 & \longrightarrow & \cdots & \longrightarrow & M_{n-1} \\
& \lrcorner & \lrcorner & \lrcorner & \lrcorner & \lrcorner & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\end{array}
\]

is a commutative diagram where

\[N_i = \text{im } f_{i-1} = \ker f_i\]

and \([0 \longrightarrow N_i \longrightarrow M_i \longrightarrow N_{i+1} \longrightarrow 0]\) is exact for \(i = 1, \ldots, n-1\).
Then, noting $\lambda(0) = 0$, 

$$\lambda(M_0) - \lambda(M_1) + \ldots + (-1)^n \lambda(M_n)$$

$$= - ( \lambda(0) - \lambda(M_0) + \lambda(N_1) )$$

$$+ ( \lambda(N_1) - \lambda(M_1) + \lambda(N_2) )$$

$$- \ldots$$

$$+ (-1)^{n-1} ( \lambda(N_n) - \lambda(M_n) + \lambda(0) )$$

$$= 0.$$