2.8 Multilinear Mappings and Tensor Products

Let $M_1, \ldots, M_r, P$ be $A$-modules.

Call a mapping $f : M_1 \times \ldots \times M_r \to P$ multilinear if it is linear in each variable, that is,

$$
f(m_1, \ldots, m_{i-1}, ax_1 + bx_2 y, m_i, \ldots, m_r) = a f(m_1, \ldots, m_{i-1}, x_1, m_i, \ldots, m_r) + b f(m_1, \ldots, m_{i-1}, x_2, m_i, \ldots, m_r)$$
Following through the proof of the main Theorem on tensor products, using \( C \) and \( D \) with \( r \)-tuples instead of ordered pairs,

we obtain a **tensor product**

\[
T = M_1 \otimes \ldots \otimes M_r,
\]

generated by products

\[
m_1 \otimes \ldots \otimes m_r = (m_1, \ldots, m_r) + D
\]

with the following properties:
**Theorem:** There exists a pair \((T, g)\) where \(T\) is an \(A\)-module and \(g : M_1 \times \ldots \times M_r \to T\) is multilinear, such that

\[
(\forall \text{ multilinear } f : M_1 \times \ldots \times M_r \to P)
\]

\[
(\exists! \text{ module homomorphism } f' : T \to P)
\]

\[M_1 \times \ldots \times M_r \xrightarrow{g} T\]

\[\xrightarrow{f} P \quad \xrightarrow{f'} \text{ commutes.}\]
Theorem continued: Moreover, if \((T, g)\) and \((T', g')\) are two pairs with this property then

\[
\exists! \text{ isomorphism } j : T \rightarrow T'
\]
We list a couple of model “canonical isomorphisms”:

**Theorem:** Let $M$, $N$, $P$ be $A$-modules. Then there exist unique isomorphisms extending the following mappings on generators:

1. $(M \otimes N) \otimes P \rightarrow M \otimes (N \otimes P) \rightarrow M \otimes N \otimes P$
   
   $$(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z) \mapsto x \otimes y \otimes z;$$

2. $(M \oplus N) \otimes P \rightarrow (M \otimes P) \oplus (N \otimes P)$
   
   $$(x, y) \otimes z \mapsto (x \otimes z, y \otimes z);$$
Proof: We prove (1) and leave

the proof of (2) as an exercise.

Fix $z \in P$ and define

$$F_z : M \times N \rightarrow M \otimes N \otimes P$$

by

$$F_z(x, y) = x \otimes y \otimes z.$$ 

Then $F_z$ is linear in the first variable, because of properties of “triple” tensors:
\[ F_z(ax_1 + bx_2, y) = (ax_1 + bx_2) \otimes y \otimes z \]

\[ = a(x_1 \otimes y \otimes z) + b(x_2 \otimes y \otimes z) \]

\[ = aF_z(x_1, y) + bF_z(x_2, y). \]

Similarly in the second variable, so \( F_z \) is bilinear.
Hence there is a unique homomorphism $f_z$ such that the following diagram commutes:

\[
\begin{array}{ccc}
M \times N & \rightarrow & M \otimes N \\
\downarrow F_z & & \downarrow f_z \\
M \otimes N \otimes P & \leftarrow & \end{array}
\]

Now consider $a, b \in A$ and $z_1, z_2 \in P$. We verify that

\[
f_{az_1 + bz_2} = af_{z_1} + bf_{z_2}.
\]
Then, for $x \in M$, $y \in N$,

$$f_{az_1+bz_2}(x \otimes y) = F_{az_1+bz_2}(x, y)$$

$$= x \otimes y \otimes (az_1 + bz_2) = a(x \otimes y \otimes z_1) + b(x \otimes y \otimes z_2)$$

$$= aF_{z_1}(x, y) + bF_{z_2}(x, y) = af_{z_1}(x \otimes y) + bf_{z_2}(x \otimes y)$$

$$= (af_{z_1} + bf_{z_2})(x \otimes y).$$
Hence \( f_{az_1 + bz_2} \) and \( af_{z_1} + bf_{z_2} \) agree on generators, so

\[
f_{az_1 + bz_2} = af_{z_1} + bf_{z_2}.
\]

Now define

\[
H : (M \otimes N) \times P \rightarrow M \otimes N \otimes P
\]

by, for \( t \in M \otimes N \) and \( z \in P \),

\[
H(t, z) = f_z(t).
\]
Then, since $f_z$ is a module homomorphism,

$$H(at_1 + bt_2, z) = f_z(at_1 + bt_2) = af_z(t_1) + bf_z(t_2)$$

$$= aH(t_1, z) + bH(t_2, z),$$

and, from the previous fact we proved,

$$H(t, az_1 + bz_2) = f_{az_1 + bz_2}(t) = (af_{z_1} + bf_{z_2})(t)$$

$$= af_{z_1}(t) + bf_{z_2}(t)$$

$$= aH(t, z_1) + bH(t, z_2).$$
Hence

\[ H \text{ is bilinear,} \]

so there is a unique homomorphism \( h \) such that the following diagram commutes:

\[
\begin{array}{ccc}
(M \otimes N) \times P & \xrightarrow{\quad} & (M \otimes N) \otimes P \\
\downarrow H & & \downarrow h \\
M \otimes N \otimes P & \xrightarrow{\quad} & 
\end{array}
\]
yielding
\[
  h\left( (x \otimes y) \otimes z \right) = H(x \otimes y, z) = f_z(x \otimes y)
  = F_z(x, y)
  = x \otimes y \otimes z .
\]

Now define
\[
  F' : M \times N \times P \rightarrow (M \otimes N) \otimes P
\]
by
\[
  F'(x, y, z) = (x \otimes y) \otimes z .
\]
It is easy to verify that $F'$ is multilinear, so there is a unique homomorphism $f'$ such that the following diagram commutes:

\[
\begin{array}{ccc}
M \times N \times P & \longrightarrow & M \otimes N \otimes P \\
\downarrow F' & & \downarrow f' \\
(M \otimes N) \otimes P & \text{whence} & (x \otimes y) \otimes z
\end{array}
\]

whence

\[
f'(x \otimes y \otimes z) = F'(x, y, z) = (x \otimes y) \otimes z.
\]
But \( h \) and \( f' \) undo each other on generators of \( M \otimes N \otimes P \) and \( (M \otimes N) \otimes P \) respectively, so

\[
h \circ f' = \text{id}_{M \otimes N \otimes P} \quad \text{and} \quad f' \circ h = \text{id}_{(M \otimes N) \otimes P}.
\]

Hence \( h \) and \( f \) are isomorphisms, and clearly \( h \) is unique with the given property.

By a similar argument there is a unique isomorphism:

\[
M \otimes (N \otimes P) \rightarrow M \otimes N \otimes P \quad x \otimes (y \otimes z) \rightarrow x \otimes y \otimes z
\]

and (1) of the Theorem is proved.