2.11 Algebras

Let $A$ be a ring.

Suppose that $B$ is both a ring and an $A$-module where the ring and module additions coincide.

Denote ring multiplication (whether in $B$ or in $A$) by juxtaposition, and scalar multiplication by elements of $A$ by $\cdot$.

(in practice both denoted by juxtaposition).
Call $B$ an $\mathbf{A}$-algebra or algebra (over $A$) if

$$(\forall a \in A) \ (\forall b, c \in B)$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \left[= b (a \cdot c) \right]$$

needed in the noncommutative case
Suppose $B$ is an $A$-algebra.

Define $f : A \to B$ by

$$f(a) = a \cdot 1 \quad (a \in A)$$

Then, for all $a_1, a_2 \in A$,

$$f(a_1 + a_2) = (a_1 + a_2) \cdot 1 = a_1 \cdot 1 + a_2 \cdot 1 = f(a_1) + f(a_2)$$
and

\[ f(a_1 a_2) = (a_1 a_2) \cdot 1 = a_1 \cdot (a_2 \cdot 1) \]

\[ = a_1 \cdot (a_2 \cdot (1 1)) = a_1 \cdot (1(a_2 \cdot 1)) \]

\[ = (a_1 \cdot 1)(a_2 \cdot 1) = f(a_1) f(a_2) , \]

which proves \( f \) is a \textbf{ring} homomorphism.
Conversely, let \( f : A \to B \) be a ring homomorphism.

Then (by restriction of scalars) \( B \) becomes an \( A \)-module by defining

\[
a \cdot b = f(a) b \quad (a \in A, \ b \in B) .
\]

Further this turns \( B \) into an \( A \)-algebra, because

\[
(\forall a \in A) \ (\forall b, c \in B)
\]

\[
a \cdot (bc) = f(a)(bc) = (f(a)b)c = (a \cdot b)c .
\]
Moreover, these processes, turning an $A$-algebra into a ring homomorphism, and vice-versa, undo each other, so

$$A$$-algebras $B$ correspond to pairs consisting of a ring $B$ and a ring homomorphism

$$f : A \rightarrow B.$$
In particular if \( A = F \) is a field and \( f : A \rightarrow B \) is not the zero homomorphism, then \( f \) is injective (an early Proposition), so \( F \) can be identified with its image under \( f \):

\[
(\forall a \in F) \quad a \equiv a \cdot 1.
\]

Thus

a nonzero \( F \)-algebra may be thought of as a ring containing \( F \) as a subring.
(2) If $A$ is any nonzero ring then

$$f : \mathbb{Z} \to A \quad \text{where} \quad f(n) = n \cdot 1$$

is easily seen to be a ring homomorphism, so $A$ becomes a $\mathbb{Z}$-algebra.

Observe that

$$\ker f = k\mathbb{Z} \quad \exists \ k \geq 0.$$ 

But $A$ is nonzero, so $k \neq 1$. We call $k$ the **characteristic** of $A$. 
We can identify

\[ \mathbb{Z}_k \ (= \mathbb{Z} \text{ if } k = 0) \]

with the subring of \( A \) generated by 1, called the prime subring of \( A \) (though it may have nothing to do with primes!).

[ If \( A \) is a field then all nonzero elements are invertible, so the prime subring must be a copy of \( \mathbb{Z} \) or a copy of \( \mathbb{Z}_p \) for some prime \( p \). ]
Algebra homomorphisms:

Let \( f : A \rightarrow B \) and \( g : A \rightarrow C \) be ring homomorphisms, so that

\[ B, C \] become \( A \)-algebras.

Consider a ring homomorphism \( h : B \rightarrow C \).

Call \( h \) an \( A \)-algebra homomorphism if \( h \) respects scalar multiplication, that is,

\[
(\forall a \in A)(\forall b \in B) \quad h(a \cdot b) = a \cdot h(b).
\]
**Observation:** $h$ is an $A$-algebra homomorphism iff the following diagram commutes:

![Diagram](https://via.placeholder.com/150)

**Proof:** $(\implies)$ If $h$ is an $A$-algebra homomorphism then

$$(\forall a \in A)(\forall b \in B) \quad h(f(a)b) = g(a)h(b),$$
so, in particular, taking \( b = 1 \),

\[
(\forall a \in A) \quad h(f(a)) = g(a),
\]

that is, \( h \circ f = g \).

\(\iff\) If \( h \circ f = g \) then, for all \( a \in A \), \( b \in B \)

\[
h(a \cdot b) = h(f(a)b) = h(f(a))h(b)
\]

\[
= g(a)h(b) = a \cdot h(b),
\]

so \( h \) is an \( A \)-algebra homomorphism.
The polynomial ring $A[t_1, \ldots, t_n]$ in $n$ commuting indeterminates is called the **free $A$-algebra** on $n$ generators because of the following:

**Property:** $A[t_1, \ldots, t_n]$ is an $A$-algebra such that if $B$ is any $A$-algebra and $b_1, \ldots, b_n \in B$ then the map

$$t_i \mapsto b_i \quad \forall i$$

extends uniquely to an $A$-algebra homomorphism: $A[t_1, \ldots, t_n] \to B$. 

If \( p(t_1, \ldots, t_n) \in A[t_1, \ldots, t_n] \) then this homomorphism is just the **evaluation** mapping

\[
p(t_1, \ldots, t_n) \mapsto p(b_1, \ldots, b_n).
\]

Evaluation is **onto** precisely when \( b_1, \ldots, b_n \) generate \( B \) as an \( A \)-algebra, in which case we say that \( B \) is **finitely generated**.

**Note:** a ring is finitely generated as a ring iff it is finitely generated as a \( \mathbb{Z} \)-algebra.
Tensor product of algebras:

Let $B$, $C$ be $A$-algebras via ring homomorphisms

$$ f : A \to B , \quad g : A \to C . $$

In particular, ignoring their ring multiplication, $B$ and $C$ become $A$-modules, so we may form the $A$-module

$$ D = B \otimes_A C . $$

We shall define multiplication on $D$ making $D$ into a ring and an $A$-algebra.
The mapping

\[ \phi : B \times C \times B \times C \rightarrow B \otimes C \]

where

\((b, c, b', c') \mapsto bb' \otimes cc' \quad (b, b' \in B, \ c, c \in C)\)

is easily checked to be multilinear.

Thus there is a unique \(A\)-module homomorphism \(\psi\) which makes the following diagram commute:
Also \textbf{(exercise)}, there is a “canonical isomorphism”

\[
\theta : (B \otimes C) \otimes (B \otimes C) \rightarrow B \otimes C \otimes B \otimes C
\]

extending the following map on generators:

\[
(b \otimes c) \otimes (b' \otimes c') \mapsto b \otimes c \otimes b' \otimes c'.
\]
Let

\[ g : (B \otimes C) \times (B \otimes C) \rightarrow (B \otimes C) \otimes (B \otimes C) \]

where

\[ (\alpha, \beta) \mapsto \alpha \otimes \beta \quad (\alpha, \beta \in B \otimes C) , \]

which is clearly bilinear.

Put

\[ \mu = \psi \circ \theta \circ g \]

so that the following diagram commutes:
\[ D \times D = (B \otimes C) \times (B \otimes C) \xrightarrow{g} (B \otimes C) \otimes (B \otimes C) \xrightarrow{\theta} B \otimes C \otimes B \otimes C \xrightarrow{\psi} B \otimes C = D \]
Further $\mu$ is bilinear (being the composite of a bilinear map with linear maps).

For $b, b' \in B$, $c, c' \in C$,

$$\mu(b \otimes c, b' \otimes c') = \psi\left(\theta\left(g(b \otimes c, b \otimes c')\right)\right)$$

$$= \psi\left(\theta((b \otimes c) \otimes (b' \otimes c'))\right)$$

$$= \psi\left(b \otimes c \otimes b' \otimes c'\right) = \phi(b, c, b', c') = (bb') \otimes (cc') .$$
The bilinearity of $\mu$ gives

$$\mu\left(\sum_i (b_i \otimes c_i), \sum_j (b'_j \otimes c'_j)\right)$$

$$= \sum_{i,j} b_i b'_j \otimes c_i c'_j .$$
It is easy now to check that $\mu$ defines a multiplication on $D = B \otimes C$ making $D$ into a ring with identity element $1 \otimes 1$. Using juxtaposition, the rule for multiplication is simply

$$
\left( \sum_i (b_i \otimes c_i) \right) \left( \sum_i (b'_j \otimes c'_j) \right) = \sum_{i,j} b_i b'_j \otimes c_i c'_j.
$$
Define $h : A \to D$ by, for $a \in A$

\[
h(a) = a \cdot (1 \otimes 1)
\]

\[
= (a \cdot 1) \otimes 1 = 1 \otimes (a \cdot 1)
\]

\[
= f(a) \otimes 1 = 1 \otimes g(a).
\]

Clearly $h$ preserves addition and maps $1$ to $1 \otimes 1$. 
Further, $h$ preserves multiplication, because, for $a_1, a_2 \in A$, 

$$h(a_1 a_2) = f(a_1 a_2) \otimes 1 = \left( f(a_1) f(a_2) \otimes (1 \ 1) \right)$$

$$= \left( f(a_1) \otimes 1 \right) \left( f(a_2) \otimes 1 \right) = h(a_1) h(a_2).$$

Thus $h$ is a ring homomorphism, with respect to which $D$ becomes an $A$-algebra.
In summary, given $A$-algebras $B$ and $C$:

The $A$-module $B \otimes_A C$ becomes an $A$-algebra with multiplication which extends the following multiplication on generators:

\[(b \otimes c)(b' \otimes c) = bb' \otimes cc'\]

for $b, b' \in B$ and $c, c' \in C$. 