3.1 Rings of Fractions

Let $A$ be a ring.

Call a subset $S$ of $A$ **multiplicatively closed** if

(i) $1 \in S$;

(ii) $(\forall x, y \in S)$ $xy \in S$.

For example, if $A$ is an integral domain then $A \setminus \{0\}$ is multiplicatively closed.
More generally, if $P$ is a prime ideal of $A$ then

$$A \setminus P \text{ is multiplicatively closed.}$$

Let $S$ be a multiplicatively closed subset of $A$. Define a relation $\equiv$ on

$$A \times S = \{(a, s) \mid a \in A, s \in S\}$$

as follows:
for \( a, b \in A \), \( s, t \in S \),

\[
(a, s) \equiv (b, t)
\]

iff \( (\exists u \in S) \ (at - bs)u = 0 \).

**Claim:** \( \equiv \) is an equivalence relation.

**Proof:** Clearly \( \equiv \) is reflexive and symmetric.

Suppose \((a, s) \equiv (b, t) \equiv (c, u)\).
Then, for some \( v, w \in S \)

\[
(at - bs)v = 0 = (bu - ct)w,
\]

so

\[
\begin{align*}
atv - bsv &= 0 \\
butw - ctw &= 0
\end{align*}
\]

so

\[
\begin{align*}
atv(uw) - bsv(uw) &= 0 \\
-ctw(sv) + butw(sv) &= 0
\end{align*}
\]
so

\[ au(tvw) - cs(tvw) = (au - cs)(tvw) = 0 , \]

But \( tvw \in S \), since \( S \) is multiplicatively closed, so \( (a, s) \equiv (c, u) \), which proves \( \equiv \) is transitive.

If \( a \in A \) and \( s \in S \) then write

\[ a/s = \text{equivalence class of } (a, s) . \]
Put

\[ S^{-1}A = \{ a/s \mid a \in A, \ s \in S \} \]

and define addition and multiplication on \( S^{-1}A \) by

\[(a/s) + (b/t) = (at + bs)/st \]
\[(a/s)(b/t) = ab/st .\]
We check that multiplication is well-defined:

Suppose

\[(a_1, s_1) \equiv (a_2, s_2) \quad \text{and} \quad (b_1, t_1) \equiv (b_2, t_2).\]

Then, for some \(u, v \in S\),

\[(a_1s_2 - a_2s_1)u = 0 \quad \text{and} \quad (b_1t_2 - b_2t_1)v = 0.\]

**WTS** \( (a_1b_1, s_1t_1) \equiv (a_2b_2, s_2t_2). \)
$$\left[(a_1 b_1)(s_2 t_2) - (a_2 b_2)(s_1 t_1)\right] uv$$


$$= (a_1 b_1)(s_2 t_2)(uv) - (a_2 b_2)(s_1 t_1)(uv) - (a_2 s_1)(b_1 t_2)(uv) + (a_2 s_1)(b_1 t_2)(uv)$$


$$= (a_1 s_2 - a_2 s_1)u(b_1 t_2v) + (b_1 t_2 - b_2 t_1)v(a_2 s_1 u)$$


$$= 0 + 0 = 0.$$
Thus

\[(a_1 b_1, s_1 t_1) \equiv (a_2 b_2, s_2 t_2),\]

which verifies that multiplication is well-defined.

**Exercise:** Prove that addition in $S^{-1}A$ is well-defined.

It is now routine to check that $S^{-1}A$ is a ring with identity $1 = s/s$ \((\forall s \in S)\).
We call $S^{-1}A$ the **ring of fractions of** $A$ **with respect to** $S$.

If $A$ is an integral domain and $S = A \setminus \{0\}$ then $S^{-1}A$ is the familiar **field of fractions** of $A$.

Let $f : A \to S^{-1}A$ where $f(x) = x/1$.

Clearly $f$ is a ring homomorphism.
Observation: If $A$ is an integral domain and $S$ any multiplicatively closed subset not containing 0 then

$$f \text{ is injective.}$$

Proof: Suppose $A$ is an integral domain, $0 \not\in S \subseteq A$, and $S$ multiplicatively closed.

Let $x_1, x_2 \in A$ such that $x_1/1 = x_2/1$. 
Then \((x_1, 1) \equiv (x_2, 1)\), so

\[(x_1 - x_2)u = 0 \quad (\exists u \in S),\]

yielding \(x_1 - x_2 = 0\), since \(A\) is an integral domain and \(u \neq 0\).

Thus \(x_1 = x_2\), proving \(f\) is injective.

If \(A\) is not an integral domain then \(f\) need not be injective:
Exercise: Let $A = \mathbb{Z}_6$, 

$$S_1 = \{ 1, 2, 4 \} \quad S_2 = \{ 1, 3 \}.$$  

Then $S_1$ and $S_2$ are multiplicatively closed.

Verify that

$$S_1^{-1}\mathbb{Z}_6 \cong \mathbb{Z}_3, \quad S_2^{-1}\mathbb{Z}_6 \cong \mathbb{Z}_2$$

(so certainly, in both cases, $f$ is not injective).

$S^{-1}A$ has the following universal property:
**Theorem:** Let \( g : A \to B \) be a ring homomorphism such that \( g(s) \) is a unit in \( B \) for each \( s \in S \).

Then there is a unique homomorphism \( h \) such that

\[
\begin{array}{ccc}
A & \xrightarrow{f} & S^{-1}A \\
\downarrow{g} & & \downarrow{h} \\
B & & \\
\end{array}
\]

commutes.
Proof: Define \( h : S^{-1}A \rightarrow B \) by

\[
h(a/s) = g(a) g(s)^{-1} \quad (a \in A, \ s \in S).
\]

WTS \( h \) is well defined.

Suppose \( a/s = a'/s' \) so \( (a, s) \equiv (a', s') \), so

\[
(as' - a's)t = 0 \quad (\exists t \in S).
\]
Thus

\[ 0 = g(0) = g((as' - a's)t) \]

\[ = [g(a)g(s') - g(a')g(s)] g(t) , \]

so, since \( g(t) \) is a unit in \( B \),

\[ g(a)g(s') - g(a')g(s) = 0 , \]

so

\[ g(a)g(s') = g(a')g(s) , \]
yielding, since \( g(s), g(s') \) are units in \( B \),

\[
g(a) g(s)^{-1} = g(a') g(s')^{-1}.
\]

This proves \( h \) is well-defined.

It is routine now to check that \( h \) is a ring homomorphism.
Further, if \( a \in A \) then

\[(h \circ f)(a) = h(a/1) = g(a)g(1)^{-1} = g(a),\]

so that the following diagram commutes:
Suppose also that \( h' : S^{-1}A \to B \) is a ring homomorphism such that the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & S^{-1}A \\
\downarrow{g} & & \downarrow{h'} \\
B & & \\
\end{array}
\]

Then

\[
h'(a/s) = h'(a/1 \cdot 1/s) = h'(a/1) h'(1/s).
\]
But $1/s$ is a unit in $S^{-1}A$ with inverse $s/1$, so that $h'(1/s)$ is a unit in $B$ and

$$h'(1/s) = [h'(s/1)]^{-1}.$$ 

Hence

$$h'(a/s) = h'(a/1) [h'(s/1)]^{-1} = h'(f(a)) [h'(f(s))]^{-1} = g(a)g(s)^{-1} = h(a/s).$$ 

This proves $h' = h$, and so $h$ is unique with the required properties.
Observe that $S^{-1} A$ and

$$f : A \to S^{-1} A, \quad a \mapsto a/1$$

have the following properties:

1. $s \in S$ implies $f(s)$ is a unit in $S^{-1} A$
   (because $s/1$ has inverse $1/s$);

2. $f(a) = 0$ implies $as = 0$ ($\exists s \in S$)
   (because the zero in $S^{-1} A$ is $0/1$);
(3) every element of $S^{-1}A$ has the form
\[ f(a)f(s)^{-1} \quad (\exists a \in A)(\exists s \in S) \]

(because $a/s = a/1 \cdot 1/s$).

Conversely, properties (1), (2), (3) characterize $S^{-1}A$ up to isomorphism:
Corollary: Let $g : A \rightarrow B$ be a ring homomorphism such that properties (1), (2) and (3) hold with $g$ replacing $f$ and $B$ replacing $S^{-1}A$.

Then there is a unique isomorphism $h$ such that the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & S^{-1}A \\
\downarrow{g} & & \downarrow{h} \\
B & \xrightarrow{h} & S^{-1}A \\
\end{array}
\]
Proof: By (1) and the previous Theorem, there is a unique homomorphism \( h : S^{-1}A \rightarrow B \) such that

\[
\begin{array}{ccc}
A & \xrightarrow{f} & S^{-1}A \\
g & \downarrow & h \\
B & \downarrow & \text{commutes.}
\end{array}
\]

Further, from the proof,

\[ h(a/s) = g(a)g(s)^{-1} \quad (a \in A, \ s \in S). \]

By (3), \( h \) is onto.
If \( a/s \in \ker h \) for some \( a \in A \), \( s \in S \), then

\[
0 = h(a/s) = g(a)g(s)^{-1},
\]

so that \( g(a) = 0 \) \( g(s) = 0 \), yielding, by (2),

\[
at = 0 \quad (\exists t \in S),
\]

whence \((a, s) \equiv (0, 1)\), that is, \( a/s = 0 \) in \( S^{-1}A \).

Thus \( h \) is one-one, so \( h \) is an isomorphism.
Examples:

(1) Let \( P \) be a prime ideal of \( A \), and put

\[
S = A \setminus P ,
\]

which is multiplicatively closed. Form

\[
A_P = S^{-1}A ,
\]

and put

\[
M = \{ a/s \in A_P \mid a \in P \} .
\]
Claim: $A_P$ is a local ring with unique maximal ideal $M$.

The process of passing from $A$ to $A_P$ is called localization at $P$.

e.g. If $A = \mathbb{Z}$ and $P = p\mathbb{Z}$ where $p$ is a prime integer, then localization at $P$ produces

$$A_P = \{ a/b \mid a, b \in \mathbb{Z}, \ p \nmid b \}.$$
Proof of Claim: We first prove

\[(\forall b \in A) \ (\forall t \in S)\]
\[b/t \in M \implies b \in P\]

Suppose
\[b/t = a/s\]
where \(b \in A\), \(a \in P\) and \(s, t \in S\). Then
\[(at - bs)u = 0 \quad (\exists u \in S)\]
so
\[ at - bs \in P \]
since \( P \) is prime, \( 0 \in P \) and \( u \not\in P \).

Hence
\[ bs = at - (at - bs) \in P. \]

But \( s \not\in P \), so \( b \in P \), and (\(*\)) is proved.

By (\(*\)), certainly \( 1 = 1/1 \not\in M \) (since \( 1 \not\in P \))
so \( M \neq A_P \).
It is easy to check that $M \triangleleft A_P$.

Further, if $b \in A$, $t \in S$ and $b/t \notin M$, then, by definition of $M$, $b \notin P$, so $b \in S$, yielding
\[ t/b \in A_P, \]
whence $b/t$ is a unit of $A_P$.

By (i) of an early Proposition (on page 105), $A_P$ is local with unique maximal ideal $M$. 
Examples (continued):

(2) \( S^{-1}A \) is the zero ring iff \( 0 \in S \).

Proof: \( \iff \) If \( 0 \in S \) then, for all \( a, b \in A \), \( s, t \in S \),

\[
a/s = b/t
\]

since

\[
(at - bs)0 = 0,
\]

so that all elements of \( S^{-1}A \) are equal.
(⇒) If $S^{-1}A$ contains only one element then

$$ (0, 1) \equiv (1, 1) $$

so that

$$ 0 = (0 \cdot 1 - 1 \cdot 1)t = -t \quad (\exists t \in S') $$

so that $0 = t \in S$. 
Let \( x \in A \) and put
\[
S = \{ x^n \mid n \geq 0 \} \quad \text{(where } x^0 = 1) .
\]

Then \( S \) is multiplicatively closed, so we may form
\[
A_x = S^{-1}A .
\]

e.g. If \( A = \mathbb{Z} \) and \( 0 \neq x \in \mathbb{Z} \) then
\[
A_x = \{ \text{rational numbers in reduced form}
\text{whose denominators divide a power of } x \} .
\]
(4) Let $I$ be an ideal of a ring $A$ and put
\[ S = 1 + I = \{ 1 + x \mid x \in I \} . \]

Then $S$ is easily seen to be multiplicatively closed, so we may form $S^{-1}A$.

e.g. If $A = \mathbb{Z}$ and $I = 6\mathbb{Z}$ then
\[ S^{-1}A = \{ \text{rational numbers in reduced form whose denominators divide some integer congruent to } 1 \mod 6 \} . \]