3.2 Modules of Fractions

Let $A$ be a ring, $S$ a multiplicatively closed subset of $A$, and $M$ an $A$-module.

Define a relation $\equiv$ on

$$M \times S = \{ (m, s) \mid m \in M, s \in S \}$$

by, for $m, m' \in M$, $s, s' \in S$. 
\[(m, s) \equiv (m', s')\]

iff \((\exists t \in S') \quad t(sm' - s'm) = 0\). 

As before it is straightforward to check that

\[\equiv\] is an equivalence relation.
If $m \in M$ and $s \in S$ then write

$$m/s = \text{equivalence class of } (m, s).$$

and put

$$S^{-1}M = \{ m/s \mid m \in M, s \in S \}.$$
Define addition and **scalar** multiplication on $S^{-1}M$ by, for $m, m' \in M$, $s, s' \in S$, $a \in A$, $t \in S$,

\[
(m/s) + (m'/s') = (sm' + s'm) / ss'
\]

\[
(a/t)(m/s) = am / ts.
\]

member of $S^{-1}A$. **
Exercise: Prove that addition and scalar multiplication are well-defined.

It is now routine to check that

\[ S^{-1}M \text{ is an } S^{-1}A\text{-module, referred to as the module of fractions with respect to } S. \]

Since the mapping \( a \mapsto a/1 \) is a ring homomorphism: \( A \to S^{-1}A \), by restriction of scalars we have
$S^{-1}M$ is an $A$-module with scalar multiplication

$$(\forall a \in A, \ m \in M, \ s \in S)$$

$$a \cdot (m/s) = (a/1)(m/s) = am/s.$$ 

The mapping: $M \to S^{-1}M$, $x \mapsto x/1$ is an $A$-module homomorphism and injective iff

$$(\forall x \in M, \ x \neq 0) \ \ Ann(Ax) \cap S = \emptyset.$$
Some notation: Let $M$ be an $A$-module.

(1) Write

$$MP = S^{-1}M$$

if $S = A\setminus P$ where $P$ is a prime ideal of $A$.

(2) Write

$$M_x = S^{-1}M$$

if $S = \{ x^n \mid n \geq 0 \}$ for some $x \in A$. 
Think of $S^{-1}$ as an “operator” which manufactures $S^{-1}A$-modules from $A$-modules.

Also $S^{-1}$ “operates” on module homomorphisms. Let $u : M \to N$ be an $A$-module homomorphism. Define

$$S^{-1}u : S^{-1}M \to S^{-1}N$$

by

$$m/s \mapsto u(m)/s \quad (m \in M, \ s \in S).$$
It is routine to check that $S^{-1}u$ is well-defined. We check $S^{-1}$ preserves addition:

\[
(S^{-1}u)\left(m_1/s_1 + m_2/s_2\right) = (S^{-1}u)\left(s_2m_1 + s_1m_2 / s_1s_2\right)
\]

\[
= u(s_2m_1 + s_1m_2) / s_1s_2 = \left[s_2 u(m_1) + s_1 u(m_2)\right] / s_1s_2
\]

\[
= u(m_1)/s_1 + u(m_2)/s_2 = (S^{-1}u)\left(m_1/s_1\right) + (S^{-1}u)\left(m_2/s_2\right).
\]

Similarly $S^{-1}$ preserves scalar multiplication.
Hence

\[ S^{-1}u \text{ is an } S^{-1}A\text{-module homomorphism} \]

(and also, by restriction of scalars, an \( A \)-module homomorphism).

Further, if

\[
\begin{align*}
&M_1 \longrightarrow M_2 \longrightarrow M_3 \\
&u \quad \quad \quad v
\end{align*}
\]

are \( A \)-module homomorphisms, then,
for all \( x \in M_1, \ s \in S, \)

\[
[S^{-1}(v \circ u)](x/s) = (v \circ u)(x) / s = v(u(x)) / s
\]

\[
= (S^{-1}v)((S^{-1}u)(x/s)) = [(S^{-1}v) \circ (S^{-1}u)](x/s),
\]

which shows

\[
S^{-1}(v \circ u) = (S^{-1}v) \circ (S^{-1}u).
\]

(We call \( S^{-1} \) a functor.)
**Theorem:** Suppose

\[ \begin{array}{ccc}
  f & g \\
  M' & \rightarrow & M \\
  \rightarrow & \rightarrow & M''
\end{array} \]

is exact at \( M \). Then

\[ \begin{array}{ccc}
  S^{-1}f & S^{-1}g \\
  S^{-1}M' & \rightarrow & S^{-1}M \\
  \rightarrow & \rightarrow & S^{-1}M''
\end{array} \]

is exact at \( S^{-1}M \).

(We call \( S^{-1} \) an **exact** functor.)
Proof: We have \( g \circ f = 0 \) the zero homomorphism, so

\[(S^{-1}g) \circ (S^{-1}f) = S^{-1}(g \circ f) = S^{-1}(0) = 0 ,\]

which proves \( \text{im} \ (S^{-1}f) \subseteq \text{ker}(S^{-1}g) \).

Suppose \( m/s \in \text{ker}(S^{-1}g) \), so \( g(m)/s \) is the zero of \( S^{-1}M'' \). Hence \( (g(m), s) \equiv (0, 1) \), so

\[0 = t g(m) = g(tm) \quad (\exists t \in S),\]

yielding \( tm \in \text{ker} g = \text{im} f \).

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Hence

\[ tm = f(m') \quad (\exists m' \in M') , \]

whence

\[ (S^{-1}f)(m'/st) = f(m')/st = tm/ts = m/s , \]

proving \( m/s \in \text{im } (S^{-1}f) . \)

Thus \( \ker(S^{-1}g) \supseteq \text{im } (S^{-1}f) , \) completing the proof of exactness at \( S^{-1}M . \)
In particular, if $M'$ is a submodule of $M$ then

$$0 \to M' \to M$$

is exact (where the mapping on the right is the inclusion embedding), so, by the Theorem,

$$0 = S^{-1}0 \to S^{-1}M' \to S^{-1}M$$

is exact, so that

inclusion induces an embedding of $S^{-1}M'$ in $S^{-1}M$.
Thus we may regard $S^{-1}M'$ as a submodule of $S^{-1}M$, identifying each $x/s$ in $S^{-1}M'$ with 
\[ x/s \text{ in } S^{-1}M' \]
with 
\[ x/s \text{ in } S^{-1}M . \]

With this identification we can prove that formation of fractions "commutes" with formation of sums, finite intersections and quotients:
**Theorem:** Let $N$ and $P$ be submodules of an $A$-module $M$. Then

(1) $S^{-1}(N + P) = (S^{-1}N) + (S^{-1}P)$;

(2) $S^{-1}(N \cap P) = (S^{-1}N) \cap (S^{-1}P)$;

(3) $S^{-1}(M/N) \cong (S^{-1}M)/(S^{-1}N)$

(as $S^{-1}A$-modules).
Proof: (1) Clearly $S^{-1}N, S^{-1}P \subseteq S^{-1}(N + P)$, so $S^{-1}N + S^{-1}P \subseteq S^{-1}(N + P)$, since $S^{-1}(N + P)$ is a submodule of $S^{-1}M$.

Also $S^{-1}(N + P) = \{ (x + y)/s \mid x \in N, y \in P, s \in S \}$

$$= \{ x/s + y/s \mid x \in N, y \in P, s \in S \}$$

$$\subseteq S^{-1}N + S^{-1}P,$$ whence equality holds.
(2) Clearly \( N \cap P \subseteq N, P \), so
\[
S^{-1}(N \cap P) \subseteq S^{-1}N, S^{-1}P,
\]
so

\[
S^{-1}(N \cap P) \subseteq S^{-1}N \cap S^{-1}P.
\]

Suppose \( \alpha \in (S^{-1}N) \cap (S^{-1}P) \), so
\[
\alpha = x/s = y/t
\]
for some \( x \in N, y \in P, s, t \in S \).

Then
\[
u(sy - tx) = 0 \quad (\exists u \in S)
\]
so

\[ usy = utx \in N \cap P . \]

Hence

\[ \alpha = x/s = (ut)x / (ut)s \in S^{-1}(N \cap P) . \]

Thus

\[ (S^{-1}N) \cap (S^{-1}P) \subseteq S^{-1}(N \cap P) , \]

whence equality holds.
(3) Observe that

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$$

is exact, where the second mapping is inclusion, and the third mapping is natural. By the previous Theorem,

$$0 \rightarrow S^{-1}N \rightarrow S^{-1}M \rightarrow S^{-1}(M/N) \rightarrow 0$$

is exact, whence

$$S^{-1}(M/N) \cong S^{-1}M / S^{-1}N.$$
**Theorem:** Let $M$ be an $A$-module. Then

$$S^{-1}M \cong S^{-1}A \otimes_A M$$

as $S^{-1}A$-modules, under the unique isomorphism

$$f : S^{-1}A \otimes_A M \to S^{-1}M$$

with the property that

$$(a/s) \otimes m \mapsto am / s \quad \ldots \quad (*)$$
Proof: Easy to see

\[ f' : S^{-1}A \times M \rightarrow S^{-1}M , \quad (a/s, m) \mapsto am / s \]

is \( A \)-bilinear, so there is a unique \( A \)-module homomorphism \( f \) making the following diagram commute yielding the rule \((\ast)\) :

\[
\begin{array}{ccc}
S^{-1}A \times M & \xrightarrow{f'} & S^{-1}A \otimes M \\
\downarrow{f'} & & \downarrow{f} \\
S^{-1}M & &
\end{array}
\]
It remains to check $f$ is bijective and preserves scalar multiplication by $S^{-1}A$.

Certainly $f$ is onto because

$$(\forall m \in M, \ s \in S) \quad m/s = f(1/s \otimes m).$$

Before proving $f$ is one-one, we prove:

**Claim:**

$$S^{-1}A \otimes M = \{ 1/s \otimes m \mid s \in S, \ m \in M \}.$$
Let

\[ \alpha = \sum_{i=1}^{n} \left( \frac{a_i}{s_i} \right) \otimes m_i \in S^{-1}A \otimes M. \]

Put

\[ s = s_1 \ldots s_n \]

and

\[ t_i = s_1 \ldots s_{i-1}s_{i+1} \ldots s_n \]

for each \( i = 1, \ldots, n \).

Then
\[ \alpha = \sum (a_i t_i / s) \otimes m_i = \sum \left[ (a_i t_i / 1) (1/s) \right] \otimes m_i \]

\[ = \sum (a_i t_i / 1) \left[ (1/s) \otimes m_i \right] = \sum (a_i t_i) \cdot \left[ (1/s) \otimes m_i \right] \]

\[ = \sum (1/s) \otimes (a_i t_i m_i) = 1/s \otimes m , \]

where \( m = \sum a_i t_i m_i \), proving the Claim.
We now prove that $f$ is one-one.

Suppose $\alpha \in \ker f$. Then, by the Claim,

$$\alpha = 1/s \otimes m \quad (\exists s \in S)(\exists m \in M),$$

so

$$0 = f(\alpha) = f(1/s \otimes m) = m/s.$$

Thus $(m, s) \equiv (0, 1)$ so

$$tm = 0 \text{ for some } t \in S.$$

Hence
\[ \alpha = \frac{1}{s} \otimes m = \frac{t}{ts} \otimes m \]

\[ = \left[ \frac{t}{1} \left( \frac{1}{st} \right) \right] \otimes m = \left( \frac{t}{1} \right) \left[ \frac{1}{st} \otimes m \right] \]

\[ = t \cdot \left[ \frac{1}{st} \otimes m \right] = \frac{1}{st} \otimes (tm) \]

\[ = \frac{1}{st} \otimes 0 = 0 . \]

Hence \( \text{ker } f = \{ 0 \} \), so \( f \) is one-one.
That \( f \) preserves scalar multiplication by elements of \( S^{-1}A \) follows from the following Lemma, whose proof is left as an exercise. This completes the proof that \( f \) is an \( S^{-1}A \)-module isomorphism.

**Lemma:** Let \( f : M_1 \rightarrow M_2 \) be an \( A \)-module homomorphism, where \( M_1 \) and \( M_2 \) are \( S^{-1}A \)-modules, regarded as \( A \)-modules by restriction of scalars.

Then \( f \) is also an \( S^{-1}A \)-module homomorphism.