3.4 Extended and Contracted Ideals in Rings of Fractions

Let $A$ be a ring and let $S$ be a multiplicatively closed subset of $A$. Throughout this section let

$$f : A \to S^{-1}A, \quad a \mapsto a/1.$$

We study extension and contraction of ideals with respect to this homomorphism.
Put

\[ \mathcal{C} = \{ I \mid I \triangleleft A \text{ and } I = I^{\text{ec}} \} , \]

the set of **contracted ideals** of \( A \) and

\[ \mathcal{E} = \{ J \mid J \triangleleft S^{-1}A \text{ and } J = J^{\text{ce}} \} , \]

the set of **extended ideals** of \( S^{-1}A \).

Consider \( I \triangleleft A \). Then \( I \) may be regarded as an \( A \)-submodule of \( A \), so we may form the module of fractions \( S^{-1}I \), and make the identification:

\[ S^{-1}I \equiv \{ a/s \in S^{-1}A \mid a \in I , \ s \in S \} . \]


Claim: \( I^e = S^{-1}I \).

\[ I^e = \langle f(I) \rangle_{\text{ideal}} \]

\[ = \left\{ \sum_{i=1}^{n} y_i f(x_i) \mid n \in \mathbb{Z}^+, \ y_i \in S^{-1}A, \ x_i \in I \ (\forall i) \right\} \]

\[ = \left\{ \sum_{i=1}^{n} \left( \frac{a_i}{s_i} \right) \left( x_i / 1 \right) \mid n \in \mathbb{Z}^+, \ a_i \in A, \ s_i \in S, \ x_i \in I \ (\forall i) \right\} . \]
Clearly, if \( a \in I \), \( s \in S \) then

\[
a/s = (1/s)(a/1) \in I^e,
\]

so \( S^{-1}I \subseteq I^e \).

Conversely, if

\[
\alpha = \sum_{i=1}^{n} (a_i/s_i)(x_i/1) \in I^e
\]

then, putting \( s = s_1 \ldots s_n \) and

\[
t_i = s_1 \ldots s_{i-1}s_{i+1} \ldots s_n \quad (\forall i),
\]
\[ \alpha = \sum \left( \frac{a_i x_i}{s_i} \right) = \sum \left( \frac{a_i x_i t_i}{s} \right) \]

\[ = \left( \sum a_i x_i t_i \right) / s \]

\[ \in S^{-1}I, \]

since \( \sum a_i x_i t_i \in I \). Thus \( I^e \subseteq S^{-1}I \), and the claim is proved.
Theorem:

(1) \( \mathcal{E} = \{ \text{all ideals of } S^{-1}A \} \).

(2) If \( I \triangleleft A \) then

\[
I^{ec} = \bigcup_{s \in S} (I : s),
\]

in which case,

\[
I^{ec} = A \iff I \cap S \neq \emptyset.
\]
Theorem continued:

(3) \( C = \{ I \triangleleft A \mid (\forall s \in S) \ s + I \) is not a zero divisor in \( A/I \} \).

(4) Prime ideals of \( S^{-1}A \) are in a one-one correspondence with prime ideals of \( A \) disjoint from \( S \), under

\[ P \mapsto S^{-1}P \quad (\forall \text{ prime ideals } P) \] .

(5) \( S^{-1} \) commutes with formation of finite sums, products, intersections and radicals.
Proof: (1) Consider \( J \triangleleft S^{-1}A \). If \( \alpha \in J \) then
\[
\alpha = x/s \quad (\exists x \in A, s \in S),
\]
so
\[
f(x) = x/1 = (s/1)(x/s) = (s/1) \alpha \in J,
\]
giving \( x \in f^{-1}(J) = J^c \), so that
\[
\alpha = (1/s)(x/1) = (1/s)f(x) \in (J^c)^e = J^{ce}.
\]
Thus \( J \subseteq J^{ce} \), and, of course, reverse set containment holds, proving all ideals of \( S^{-1}A \) are extended.
Suppose \( I \triangleleft A \). Then, for all \( x \in A \),

\[
\begin{align*}
    x & \in I^{ec} \iff x \in (S^{-1}I)^c \quad \text{(by the Claim)} \\
    \iff x/1 & \in S^{-1}I \\
    \iff x/1 & = a/s \quad (\exists a \in I, s \in S) \\
    \iff (xs - a)t & = 0 \quad (\exists a \in I, s, t \in S)
\end{align*}
\]
\[ \iff \quad xst = at \quad (\exists a \in I, \ s, t \in S) \]

\[ \iff \quad xu \in I \quad (\exists u \in S) \]

(since \( S \) is multiplicatively closed)

\[ \iff \quad x \in \bigcup_{u \in S} (I : u) . \]
Hence

\[ I^{ec} = \bigcup_{u \in S} (I : u), \]

and thus

\[ I^{ec} = A \iff A = \bigcup_{u \in S} (I : u) \]

\[ \iff u = 1u \in I \quad (\exists u \in S) \]

\[ \iff I \cap S \neq \emptyset. \]
(3) If $I \triangleleft A$ then

$$I \in C \iff I = I^{ec} \iff I \supseteq I^{ec}$$

$$\iff I \supseteq (S^{-1}I)^c$$

(by the earlier Claim)

$$\iff f^{-1}(S^{-1}I) \subseteq I$$
\[
\Longleftrightarrow (\forall x \in I, s \in S, y \in A)
\]

\[
x/s = y/1 \quad \Longrightarrow \quad y \in I
\]

\[
\Longleftrightarrow (\forall x \in I, s \in S, y \in A)
\]

\[
(\exists t \in S) \ (x - ys)t = 0 \quad \Longrightarrow \quad y \in I
\]
\[
\iff \ (\forall u \in S \ , \ y \in A) \quad yu \in I \implies y \in I
\]

\[
\iff \ (\forall u \in S \ , \ y \in A)
\]

\[
(y + I)(u + I) = I \implies y + I = I
\]

\[
\iff \ (\forall u \in S) \quad u + I \text{ is not a zero divisor of } A/I,
\]

which proves that \((3)\) holds.
(4) Let

\[ \mathcal{P}_1 = \left\{ P \triangleleft A \mid P \text{ prime }, \ P \cap S = \emptyset \right\}, \]

\[ \mathcal{P}_2 = \left\{ Q \triangleleft S^{-1}A \mid Q \text{ prime } \right\}, \]

and

\[ \Phi : \mathcal{P}_1 \rightarrow \mathcal{P}_2 \]

where

\[ \Phi(P) = P^e = S^{-1}P \quad (\forall P \in \mathcal{P}_1). \]

WTS \ \Phi \ is sensibly defined.
Let $P \in \mathcal{P}_1$. Certainly $S^{-1}P \triangleleft S^{-1}A$.

Suppose

$$\alpha, \beta \in S^{-1}A \quad \text{and} \quad \alpha \beta \in S^{-1}P.$$  

WTS $\alpha \in S^{-1}P$ or $\beta \in S^{-1}P$.

Now

$$\alpha = a/s, \beta = b/t \quad (\exists a, b \in A, s, t \in S).$$
Then
\[ ab / st = \alpha \beta = c/u \ (\exists c \in P, u \in S), \]
so
\[ (abu - cst)v = 0 \ (\exists v \in S), \]
yielding
\[ abuv = cstv \in P. \]
But \( P \) is prime and \( u, v \notin P \), so either
\[ a \in P \quad \text{or} \quad b \in P , \]
which proves
\[ \alpha = a/s \in S^{-1}P \quad \text{or} \quad \beta = b/t \in S^{-1}P. \]
Thus \( S^{-1}P \) is prime, so that \( \Phi \) is sensibly defined.

\[
\text{WTS } \Phi \text{ is one-one.}
\]

Suppose \( P_1, P_2 \in \mathcal{P}_1 \) and \( \Phi(P_1) = \Phi(P_2) \), that is,
\[ S^{-1}P_1 = S^{-1}P_2. \]
If \( x \in P_1 \) then

\[
x/1 = y/s \quad (\exists y \in P_2, s \in S),
\]

so

\[
(xs - y)t = 0 \quad (\exists t \in S)
\]

yielding

\[
xs \tau = yt \in P_2
\]

whence \( x \in P_2 \) (since \( s, t \notin P_2 \)). This shows \( P_1 \subseteq P_2 \) and similarly \( P_2 \subseteq P_1 \), whence equality. Thus \( \Phi \) is one-one.
WTS $\Phi$ is onto.

Suppose $Q \in \mathcal{P}_2$, so

$$Q^c = f^{-1}(Q) \triangleleft A$$

is prime, being the preimage of a prime ideal by a ring homomorphism.

By (1), $Q^{ce} = Q$. If $s \in Q^c \cap S$ then

$$1/1 = (1/s)(s/1) = (1/s)f(s) \in Q^{ce} = Q,$$

so $Q = S^{-1}A$, contradicting that $Q$ is prime.
Hence $Q^c \cap S = \emptyset$, so

$$Q^c \in P_1$$

and

$$\Phi(Q^c) = Q^{ce} = Q.$$  

Thus $\Phi$ is onto, and the proof of (4) is complete.

(5) Let $I_1, I_2 \in A$. Then, by earlier exercises,

$$S^{-1}(I_1 + I_2) = (I_1 + I_2)^e = I_1^e + I_2^e$$

$$= S^{-1}I_1 + S^{-1}I_2,$$
and

\[ S^{-1}(I_1 I_2) = (I_1 I_2)^e = I_1^e I_2^e \]

\[ = (S^{-1}I_1)(S^{-1}I_2) . \]

Further, regarding \( I_1, I_2 \) as \( A \)-modules,

\[ S^{-1}(I_1 \cap I_2) = (S^{-1}I_1) \cap (S^{-1}I_2) , \]

by a recently proved theorem.
Suppose $I \triangleleft A$. Then, by an early exercise,

$$S^{-1} r(I) = [r(I)]^e \subseteq r(I^e) = r(S^{-1}I).$$

If $\alpha \in r(S^{-1}I)$ then

$$\alpha^n \in S^{-1}I \quad (\exists n \geq 1)$$

so that, writing $\alpha = a/s \quad (\exists a \in A, \ s \in S)$,

$$a^n/s^n = b/t \quad (\exists b \in I)(\exists t \in S)$$
so

\[(a^n t - b s^n)u = 0 \quad (\exists u \in S)\]

so

\[(atu)^n = (a^n t u)(t^{n-1} u^{n-1}) = (b s^n u) t^{n-1} u^{n-1} \in I,\]

so \(atu \in r(I)\) and

\[\alpha = a/s = (atu)/(stu) \in S^{-1}(r(I)).\]

Thus \(r(S^{-1}I) \subseteq S^{-1}(r(I))\), so equality holds, finally completing the proof of the Theorem.
Corollary: The nilradical of $S^{-1}A$ is $S^{-1}N$ where $N$ is the nilradical of $A$.

Proof: By (5) of the previous Theorem,

$$S^{-1}N = S^{-1}(r(\{0\})) = r(S^{-1}\{0\})$$

$$= r(\{0/1\}),$$

which is the nilradical of $S^{-1}A$. 
**Corollary:** Let $P$ be a prime ideal of $A$.

Then the prime ideals of the local ring $A_P$ are in a one-one correspondence with the prime ideals of $A$ contained in $P$.

**Proof:** By part (4) of the previous Theorem this correspondence arises, because

an ideal avoids $S = A \setminus P$

iff

the ideal is contained in $P$. 636
These considerations tell us that

constructing $A_P$ focuses attention on prime ideals contained in $P$.

On the other hand

constructing $A/P$ focuses attention on prime ideals containing $P$. 
Suppose also that \( Q \) is a prime ideal and \( P \supseteq Q \).

To focus attention on prime ideals \textbf{between} \( P \) and \( Q \) suggests constructing the hybrid

\[
S^{-1}A / S^{-1}Q \cong T^{-1}(A/Q) \quad \cdots (*)
\]

(isomorphism proved below)

where \( S = A \setminus P \) and \( T = (A/Q) \setminus (P/Q) \).
In particular, if \( P = Q \) then

\[
S^{-1}Q = \{ \frac{x}{s} \mid x \in P, \ s \in S \}
\]

is the unique maximal ideal of \( A_P = S^{-1}A \) and

\[
T = \{ \text{nonzero elements of } A/P \}
\]

so (\( \ast \)) becomes
residue field of \( A_P \)

\( \cong \) field of fractions of the integral domain \( A/P \).

**Proof of (\( \ast \))**:

Let \( \phi : S^{-1}A \rightarrow T^{-1}(A/Q) \) where

\[
\frac{a}{s} \mapsto \frac{a + Q}{s + Q} \quad (a \in A, \ s \in S).
\]

It is easy to check that \( \phi \) is a well-defined, onto...
ring homomorphism.

Clearly \( S^{-1}Q \subseteq \ker \phi \).

Suppose \( a/s \in \ker \phi \). Then

\[
a + Q / s + Q = Q / 1 + Q,
\]

so

\[
(a + Q)(t + Q) = Q \quad (\exists t \in S).
\]

Hence

\[
Q = at + Q \quad \text{yielding} \quad at \in Q.
\]
But $Q$ is prime and $t \notin P \supseteq Q$.

Thus $a \in Q$, so

$$a/s \in S^{-1}Q,$$

which proves $\ker \phi = S^{-1}Q$.

The result (*) now follows by the Fundamental Homomorphism Theorem.