4.3 Composition Series

Let $M$ be an $A$-module.

A series for $M$ is a strictly decreasing sequence of submodules

$$M = M_0 \supset M_1 \supset \ldots \supset M_n = \{0\}$$

beginning with $M$ and finishing with $\{0\}$.

The length of this series is $n$. 
A composition series is a series in which no further submodule can be inserted which, for the above, is equivalent to saying each composition factor $M_i/M_{i+1}$ is simple, that is, each $M_i/M_{i+1}$ is nontrivial and has no submodule except for itself and the trivial submodule.
Example: The $\mathbb{Z}$-module $\mathbb{Z}_{30}$ has the following lattice of submodules:

$$\mathbb{Z}_{30} = \langle 1 \rangle$$

$$\{0\} = \langle 0 \rangle$$
Any path from the top to the bottom will yield a composition series:

e.g. $\langle 1 \rangle \supset \langle 2 \rangle \supset \langle 6 \rangle \supset \langle 0 \rangle$,

with composition factors:

$$\langle 1 \rangle/\langle 2 \rangle = \mathbb{Z}_{30}/2\mathbb{Z}_{30} = (\mathbb{Z}/30\mathbb{Z})/(2\mathbb{Z}/30\mathbb{Z})$$

$$\cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2,$$
\[ \langle 2 \rangle / \langle 6 \rangle \cong 2\mathbb{Z} / 6\mathbb{Z} \cong \mathbb{Z}_3 , \]

\[ \langle 6 \rangle / \langle 0 \rangle \cong 6\mathbb{Z} / 30\mathbb{Z} \cong \mathbb{Z}_5 \]

In fact, all composition series for \( \mathbb{Z}_{30} \) produce composition factors \( \mathbb{Z}_2 , \mathbb{Z}_3 , \mathbb{Z}_5 \) (in accordance with the Jordan-Holder Theorem below).
Notation: If $N$ is a module, let $\ell(N)$ denote the least length of a composition series of $N$, if one exists, and put $\ell(N) = \infty$ if no composition series for $N$ exists.

**Theorem:** Suppose $M$ has a composition series of length $n$.

Then every composition series of $M$ has length $n$, and every series can be refined (that is, submodules can be inserted) to yield a composition series.
Proof: Here $\ell(M) \leq n$. Consider a submodule $N$ of $M$ where $N \neq M$. We first show

$$\ell(N) < \ell(M).$$  

(\textcolor{red}{\ast})

Put $\ell = \ell(M)$ and let

$$M = M_0 \supset M_1 \supset \ldots \supset M_\ell = \{0\}$$

be a composition series of length $\ell$. Then

$$N = N \cap M_0 \supseteq N \cap M_1 \supseteq \ldots \supseteq N \cap M_\ell = \{0\}$$

is a chain of submodules from $N$ to $\{0\}$. 

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By a module isomorphism theorem, for each \( i \),

\[
\frac{N \cap M_i}{N \cap M_{i+1}} = \frac{N \cap M_i}{(N \cap M_i) \cap M_{i+1}}
\]

\[
\Leftrightarrow \frac{(N \cap M_i) + M_{i+1}}{M_{i+1}}
\]

the last of which is a submodule of the simple module \( M_i / M_{i+1} \). Hence

\[
\frac{N \cap M_i}{N \cap M_{i+1}} \text{ is trivial or simple.}
\]
Thus, deleting repetitions from the above chain from \( N \) to \( \{0\} \) must yield a composition series for \( N \), which proves

\[
\ell(N) \leq \ell(M).
\]

Suppose \( \ell(N) = \ell \).

Then, no repetitions occurred in the previous process, so, by the earlier isomorphism,

\[
(N \cap M_i) + M_{i+1} = M_i \quad (\forall i).
\]
We observe, by induction, that

\[
N \cap M_i = M_i \quad (\forall i),
\]

since, clearly

\[
N \cap M_\ell = N \cap \{0\} = \{0\} = M_\ell,
\]

which starts the induction,

and, for \( i \leq \ell - 1 \), using \((**\)**) and an inductive
hypothesis,

\[ N \cap M_i = (N \cap M_i) + (N \cap M_{i+1}) \]

\[ = (N \cap M_i) + M_{i+1} = M_i. \]

In particular,

\[ M = M_0 = N \cap M_0 = N \cap M = N, \]

which contradicts that \( N \neq M \).

Hence \( \ell(N) < \ell \) and \((*)\) is proved.
Now consider any series

\[ M = M'_0 \supset M'_1 \supset \ldots \supset M'_k = \{0\} \quad (\dagger) \]

of length \( k \). By (\(*\)),

\[ \ell = \ell(M) > \ell(M'_1) > \ldots > \ell(M'_k) = 0, \]

so

\[ \ell \geq k. \]

If (\(\dagger\)) is a composition series, then \( \ell \leq k \), by definition of \( \ell \), so \( \ell = k \).
This proves

all composition series of $M$ have the same length $n$.

If $(\dagger)$ is not a composition series, then $k < n$, because if $k = n$, then we can insert another module somewhere to get another series of length $n + 1 \leq \ell = n$, which is nonsense.

Hence any series which is not a composition series can be successively refined until its length is $n$, in
which case it becomes a composition series, and the
Theorem is proved.

\begin{frame}
\begin{corollary}
A module $M$ has a composition series iff $M$ satisfies the a.c.c. and d.c.c.
\end{corollary}
\end{frame}

\textbf{Proof:} \hspace{1em} (\implies) \hspace{1em} If $M$ has a composition series of length $n$, then, by the previous Theorem,
\begin{align*}
\text{all series have length } \leq n,
\end{align*}
so all ascending and descending chains must be stationary,

that is, $M$ satisfies both the a.c.c. and the d.c.c.

$(\iff)$ Suppose $M$ satisfies both the a.c.c. and the d.c.c.

If $M = \{0\}$ then certainly $M$ has a composition series.

Suppose $M \neq \{0\}$. 
Since $M$ satisfies the maximal condition (equivalent to the a.c.c),

the set of proper submodules of $M$ has a maximal element $M_1$, so $M/M_1$ is simple.

Suppose we have found submodules

$$M = M_0 \supset M_1 \supset \ldots \supset M_k \quad (k \geq 1)$$

where $M_i/M_{i+1}$ is simple for $i = 0, \ldots, k - 1$. 
If $M_k = \{0\}$ then we have a composition series for $M$.

If $M_k \neq \{0\}$, then, again since $M$ satisfies the maximal condition, we can find a submodule $M_{k+1}$ of $M_k$ such that $M_k/M_{k+1}$ is simple.

Either $M$ has a composition series, or by induction we have an infinite strictly descending chain

$$M = M_0 \supset M_1 \supset \ldots \supset M_k \supset \ldots$$
The latter is excluded because $M$ satisfies the d.c.c.

Hence $M$ has a composition series and the Corollary is proved.

Say that a module $M$ has **finite length** if it has a composition series

(equivalently satisfies both the a.c.c. and d.c.c.)

in which case all composition series have the same length $\ell(M)$, called the **length** of $M$. 
We now prove a uniqueness result concerning the composition factors:

**Jordan-Holder Theorem:** There is a one-one correspondence between the composition factors of any two composition series of a module of finite length such that corresponding factors are isomorphic.

**Proof:** Let $M$ be a module of finite length $\ell$. 
If $\ell = 0$ then the set of composition factors is always empty, so the result is vacuously true, which starts an induction.

Suppose $\ell > 0$ and let

$$M = M_0 \supset M_1 \supset \ldots \supset M_\ell = \{0\}$$

$$M' = M'_0 \supset M'_1 \supset \ldots \supset M'_\ell = \{0\}$$

be two composition series.
If $M_1 = M'_1$ then, by an inductive hypothesis

\[(\text{since } \ell(M_1) = \ell - 1),\]

there is an appropriate correspondence between

\[\{ M_i/M_{i+1} \mid i = 1, \ldots, \ell - 1 \} \]

and

\[\{ M'_i/M'_{i+1} \mid i = 1, \ldots, \ell - 1 \} ,\]

so, since

\[M_0/M_1 = M'_0/M'_1 ,\]
there is an appropriate correspondence between

\[
\{ \frac{M_i}{M_{i+1}} \mid i = 0, \ldots, \ell - 1 \}
\]

and

\[
\{ \frac{M'_i}{M'_{i+1}} \mid i = 0, \ldots, \ell - 1 \},
\]

and we are done.

Suppose then \( M_1 \neq M'_1 \), so \( M_1 + M'_1 = M \), since \( M_1 \) is maximal in \( M \).
\[ M = M_1 + M'_1 \]
By module isomorphism theorems,

\[ M/M_1 = M_1 + M'_1 / M_1 \cong M'_1 / M_1 \cap M'_1 \]

and

\[ M/M'_1 = M_1 + M'_1 / M'_1 \cong M_1 / M_1 \cap M'_1 . \]

But

\[ \ell(M_1 \cap M'_1) < \ell(M_1), \ell(M_2) < \ell \]

so we can apply an inductive hypothesis to composition series of these modules.
Let $F_1, F_2, F_1'$ be the collections of composition factors for $M_1, M_1 \cap M_1', M_1'$ respectively. Then there are appropriate correspondences between

\[ F_1 \cup \{M/M_1\} \]

and \[ F_2 \cup \{M_1/M_1 \cap M_1', M/M_1\} \]

and \[ F_2 \cup \{M/M_1', M_1'/M_1 \cap M_1'\} \]

and \[ F_1' \cup \{M/M_1'\} \]

which proves the Theorem.
**Theorem:** The length $\ell(M)$ of a module $M$ defines an additive function on the class of all $A$-modules of finite length.

**Proof:** Suppose

$$
\begin{array}{ccc}
\alpha & \beta \\
0 & \longrightarrow & M' \\
& \longrightarrow & M \\
& \longrightarrow & M'' \\
& \longrightarrow & 0
\end{array}
$$

is exact, where all modules have finite length. We need to show

$$
\ell(M) = \ell(M') + \ell(M'').
$$
Let

\[ M' = M'_0 \supset M'_1 \supset \ldots \supset M'_{\ell(M')} = \{0\} \]

\[ M'' = M''_0 \supset M''_1 \supset \ldots \supset M''_{\ell(M'')} = \{0\} \]

be composition series for \( M' \) and \( M'' \) respectively. Since \( \alpha \) is injective,

\[ \alpha(M') = \alpha(M'_0) \supset \alpha(M'_1) \supset \ldots \supset \alpha(M'_{\ell(M')}) = \{0\} \]

is a composition series for \( \alpha(M') = \ker \beta \).
Since $\beta$ is onto, 

$$M/\ker \beta = \beta^{-1}(M''_0)/\ker \beta$$

$$\supset \beta^{-1}(M''_1)/\ker \beta \supset$$

$$\ldots \supset \beta^{-1}(M''_{\ell(M''_\ell)})/\ker \beta = \{\ker \beta\}$$

is a composition series for $M/\ker \beta$. Combining these two series produces the following series for $M$:
\[ M = \beta^{-1}(M_0'') \supset \beta^{-1}(M_{1''}) \supset \ldots \supset \beta^{-1}(M_{\ell(M'')}) = \ker \beta \]

\[{0} = \alpha(M'_{\ell(M')}) \subset \ldots \subset \alpha(M'_{0}) = \alpha(M') \]

But this is a composition series, because, by another isomorphism theorem, for \( i = 0 , \ldots , \ell(M'') - 1 \),
\[
\beta^{-1}(M''_i) / \beta^{-1}(M''_{i+1})
\]
\[
\cong \left( \beta^{-1}(M''_i) / \ker \beta \right) / \left( \beta^{-1}(M''_{i+1}) / \ker \beta \right)
\]

which is simple. Hence

\[
\ell(M) = \ell(M') + \ell(M''),
\]

and the Theorem is proved.
Interpreting the theory for vector spaces yields:

**Proposition:** Let $V$ be a vector space over a field $F$. TFAE

(i) $V$ is finite dimensional.
(ii) $V$ has finite length.
(iii) $V$ satisfies the a.c.c.
(iv) $V$ satisfies the d.c.c.

If any of these hold, then dimension equals length.
Proof: (i) $\iff$ (ii) If $\{x_1, \ldots, x_n\}$ is a basis for $V$ then

$$V = \langle x_1, \ldots, x_n \rangle \supset \langle x_1, \ldots, x_{n-1} \rangle \supset \cdots \supset \langle x_1 \rangle \supset \{0\}$$

is a composition series of length $n$.

(ii) $\implies$ (iii), (ii) $\implies$ (iv): follow from the earlier Corollary on page 781.
(iii) $\Rightarrow$ (i), (iv) $\Rightarrow$ (i): Suppose (i) is false, so $V$ contains an infinite sequence of linearly independent vectors. For each $n \geq 1$ put

$$U_n = \langle x_1, \ldots, x_n \rangle,$$

$$V_n = \langle x_{n+1}, x_{n+2}, \ldots \rangle.$$

Then
\{0\} \subset U_1 \subset U_2 \subset \ldots \subset U_n \subset \ldots

V \supset V_1 \supset V_2 \supset \ldots \supset V_n \supset \ldots

are strictly ascending and descending chains respectively,

so that both (iii) and (iv) fail to hold.
Corollary: Let $A$ be a ring in which

$$M_1 \cdot M_2 \cdots M_N = \{0\}$$

for some (not necessarily distinct) maximal ideals $M_1, \ldots, M_n$.

Then $A$ is Noetherian iff $A$ is Artinian.
Proof: First note that if

\[ I \trianglelefteq A, \ M \trianglelefteq A, \ M \text{ maximal} \]

then \( I / IM \) is an \( A \)-module, so also

\[ I/IM \text{ is an } A/M\text{-module} \]

(since \( M \subseteq \text{Ann}(I/IM) \)), that is,

\[ I/IM \text{ is a vector space over the field } A/M, \]
so, by the previous Proposition,

\[ I/IM \] satisfies the a.c.c. on subspaces (ideals)

iff

\[ I/IM \] satisfies the d.c.c. on subspaces (ideals).

Consider the chain

\[ A \supset M_1 \supseteq M_1M_2 \supseteq \ldots \supseteq M_1 \ldots M_n = \{0\} . \]

Then, by repeated application of an earlier Theorem about exactness (on page 756),
\(A\) satisfies the a.c.c. on ideals

iff each factor

\[
A/M_1, M_1/M_1M_2, \ldots, M_1\ldots M_{n-1} / M_1\ldots M_n
\]

satisfies the a.c.c. on ideals

iff

each factor satisfies the d.c.c. on ideals

iff

\(A\) satisfies the d.c.c. on ideals,

and the Corollary is proved.