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OPTIMISATION

In an optimisation problem we are given some function, say $f(x)$, and are required to find values of x for which $f(x)$ has an extreme value locally. An extreme value is either

(1) a maximum value or
(2) a minimum value.

Graphically this corresponds to finding the highest or lowest points, respectively, on the graph $y = f(x)$.

Optimisation is an area of mathematics where calculus proves both useful and elegant.

We proceed by considering some motivating optimisation problems.

In the context of optimisation problems calculus has been employed in problems of biology, engineering, physics, financial maths, agriculture, infectious diseases and many other applications.

Example 1:

Consider the graph $y = f(x)$ where $f(x) = x^3 - 3x^2 + 1$

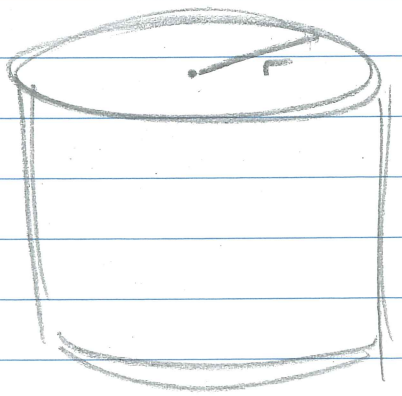
(I) We wish to find the local maximum and minimum points on this curve.

(II) We wish to graph this curve.

Example 2:

Consider a container of Maltesers. Given $600\ cm^2$ of material we wish to construct a container of maximum volume.

(I) Determine the height & radius of the cylindrical container which maximizes the volume of the container made with the material given.



Volume $= \pi r^2 h$

Surface area $= 2\pi r^2 + 2\pi r h$.

Recall that, by calculus, we obtain the following information.

Given $y = f(x)$, derive once to obtain $\frac{dy}{dx} = f'(x)$. Derive again to obtain $\frac{d^2y}{dx^2} = f''(x)$.

$y = f(x)$

if $f(x) > 0$, $f(x)$ is above the x-axis.
if $f(x) = 0$, $(x, f(x))$ is an x-intercept.
if $f(x) < 0$, $f(x)$ is below the x-axis.

$\frac{dy}{dx} = f'(x)$

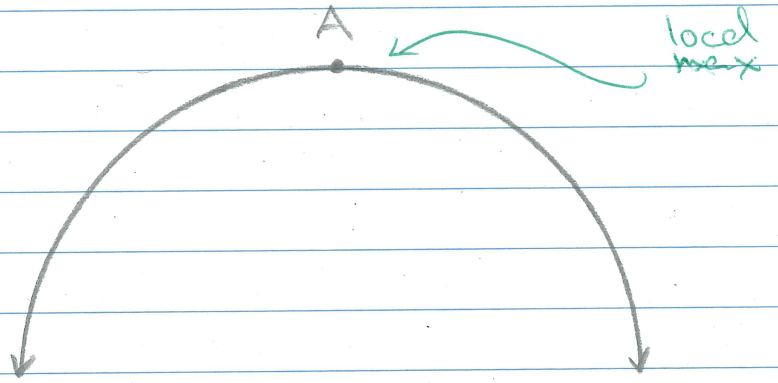
if $f'(x) > 0$, $f(x)$ is increasing. ↗
if $f'(x) = 0$, $(x, f(x))$ is a stationary point.
if $f'(x) < 0$, $f(x)$ is decreasing. ↘

$\frac{d^2y}{dx^2} = f''(x)$

if $f''(x) > 0$, $f(x)$ is concave up. ∪
if $f''(x) = 0$, $(x, f(x))$ is a possible inflection point.
if $f''(x) < 0$, $f(x)$ is concave down. ∩

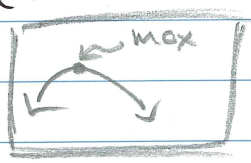
LOCAL MAXIMUM & MINIMUM POINTS

Consider a section of a graph with the following structure



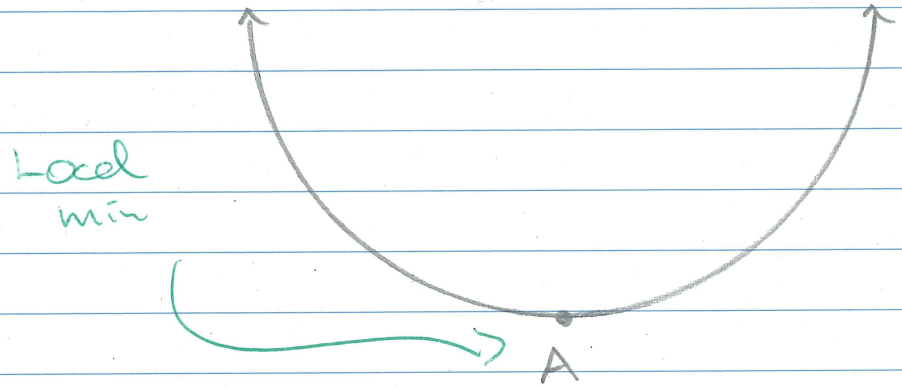
Positive slope	Zero slope	Negative slope
$f'(x) > 0$	$f'(x) = 0$	$f'(x) < 0$

Note that this local structure has the basic structure of a concave down curve, ie., where $f''(x) < 0$ locally.



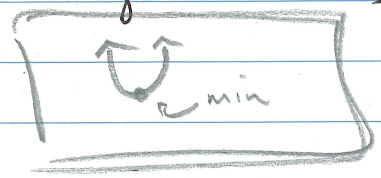
The local structure indicates that the point A is a LOCAL MAXIMUM.

Consider a section of a graph with the following structure.



Negative slope	zero slope	Positive slope
$f'(x)<0$	$f'(x)=0$	$f'(x)>0$.

Note that this local structure has the basic structure of a concave up curve, i.e., where $f''(x)>0$ locally.



The local structure indicates that the point A is a LOCAL MINIMUM.

FINDING & CLASSIFYING STATIONARY POINTS

To find and classify stationary points or to find maximum/minimum points there are 3 basic steps.

STEP 1 (DERIVE)

Derive $y = f(x)$ to obtain the first and second derivatives.

$y = f(x)$

$\frac{dy}{dx} = f'(x)$

$\frac{d^2y}{dx^2} = f''(x)$.

← GET THESE!

Make sure they are in a fairly simplified form.

STEP 2 (FIND STATIONARY POINTS)

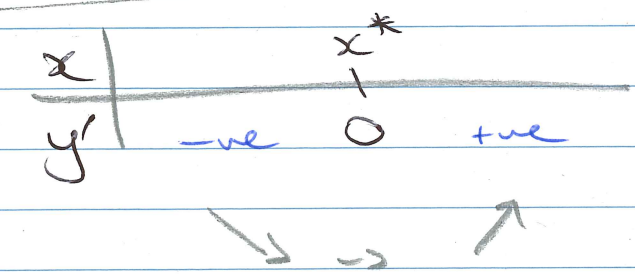
Find the values of x for which $f'(x) = 0$. Sub these x-values into $y = f(x)$ to find the corresponding y-values. The resulting points are stationary points.

(Remember, both the local max & min points have $f'(x) = 0$! That is, to find max/min look for $f'(x) = 0$!)

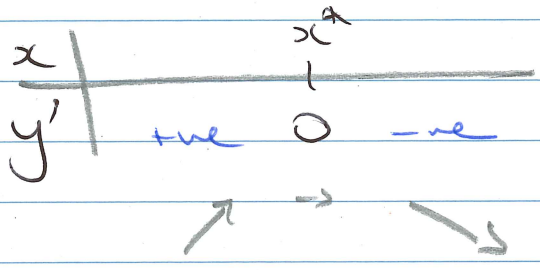
STEP 3: (CLASSIFICATION)

This can be done using either of 2 methods.

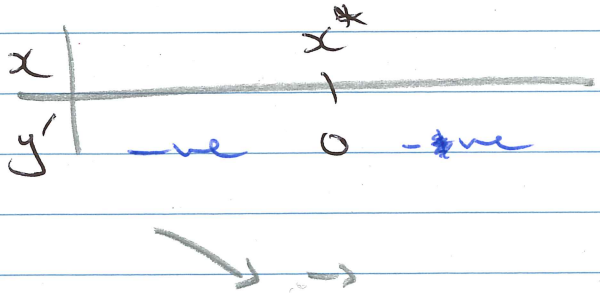
a) First derivative sign diagram



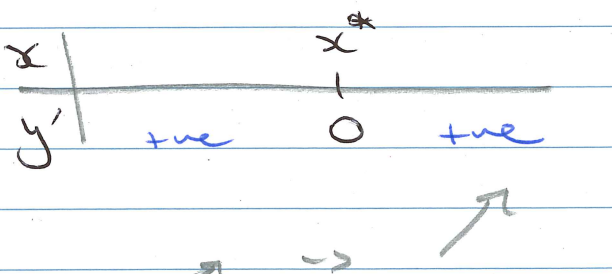
⇒ MINIMUM



⇒ MAXIMUM



⇒ horizontal inflection



⇒ horizontal inflection

This method is longer than the second (most times) but always provides a conclusive result in classification.

b) Second derivative test

Simply determine the sign of the second derivative at the x-value where $f'(x) = 0$.

if $f''(x) > 0 \Rightarrow$ ∪ MINIMUM

if $f''(x) < 0 \Rightarrow$ ∩ MAXIMUM

if $f''(x) = 0 \Rightarrow$ INCONCLUSIVE!

Must use first derivative sign diagram.

(Possible inflection points)

For many functions this is a fast way of classifying max/min points.

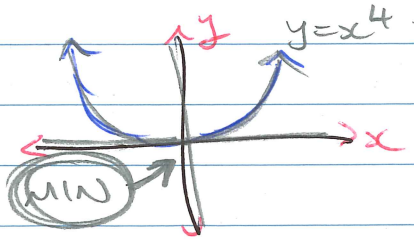
Note, if finding the second derivative is very difficult or time consuming this method is useful, but may not give a conclusive result in classification.

Example

$y = x^4$

$y' = 4x^3$

$y'' = 12x^2$



stat. pt. @ $x = 0$ $(y' = 0)$. $y'' = 0$ at $x = 0 \Rightarrow$ INCONCLUSIVE!

Motivating Example 1:

$$y = x^3 - 3x^2 + 1$$

Question: (1)(I)
find max/min points.

Step 1: (Derivatives)

$$y = x^3 - 3x^2 + 1$$

$$\frac{dy}{dx} = 3x^2 - 6x$$

$$\frac{d^2y}{dx^2} = 6x - 6.$$

Step 2: (find stationary pts).

$\frac{dy}{dx} = 0$ if $3x^2 - 6x = 0$.

$$3x(x-2) = 0.$$

ie $x = 0$
or $x = 2$.

Stationary points

if $x = 0$ $\quad y = (0)^3 - 3(0)^2 + 1 = 1$. $\quad A(0, 1)$

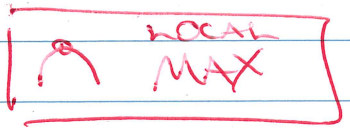
if $x = 2$ $\quad y = 2^3 - 3(2)^2 + 1 = 8 - 12 + 1 = -3$ $\quad B(2, -3)$

Step 3) (classification)

using the second derivative test,

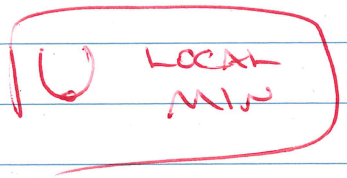
at $A(0,1)$

$$\frac{d^2y}{dx^2} = 6\times 0 - 6$$
$$= -6 < 0 \Rightarrow$$



at $B(2,-3)$

$$\frac{d^2y}{dx^2} = 6\times 2 - 6 \Rightarrow$$
$$= 6 > 0$$

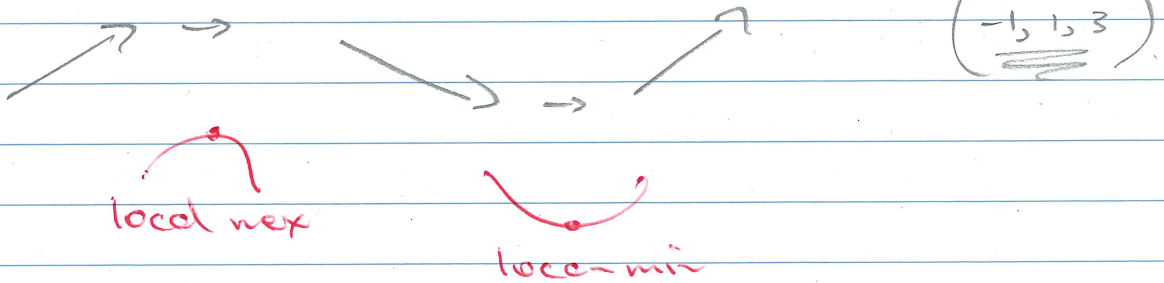


OR

using the first derivative sign diagram.

$y' = 3x^2 - 6x$

x		0		2	
y'	+ve	0	−ve	0	+ve



Thus we have found that

A(0,1) is a local maximum point

and

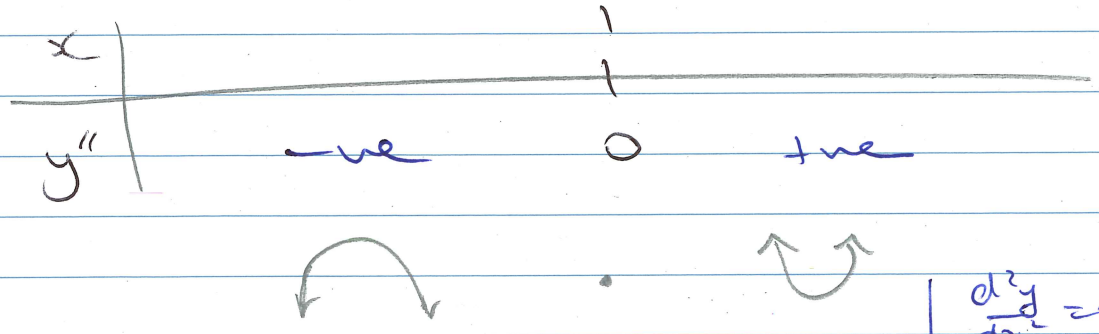
B(2,-3) is a local minimum point.

In order to sketch $y = x^3 - 3x^2 + 1$ we need only to determine the concavity along the graph.

Question: (I) (II)
Sketch
$y = x^3 - 3x^2 + 1$

We use a second derivative sign diagram & find any points of inflection.

$y'' = 6x - 6$



There is a change in concavity!

if $x = 1$, $y = 1^3 - 3(1)^2 + 1$
$= -1$

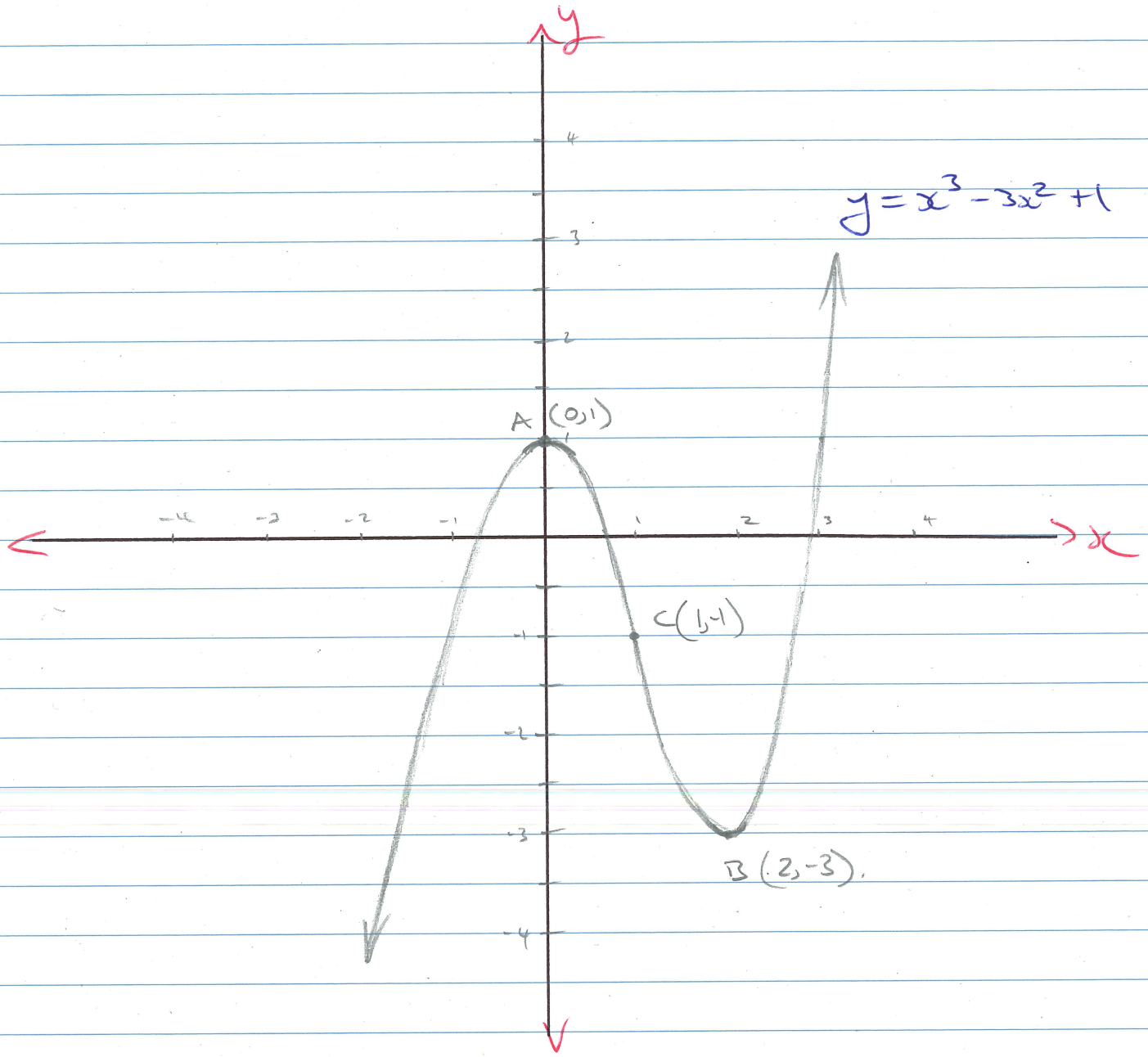
C(1,-1) is a point of inflection.

$\frac{d^2y}{dx^2} = 0$ if

$6x - 6 = 0$

$x = 1$

test $x = 0, 2$.


y
x
$y = x^3 - 3x^2 + 1$
A (0,1)
C(1,-1)
B (2,-3).
4
3
2
-1
-2
-3
-4
-2
-1
1
2
3
4