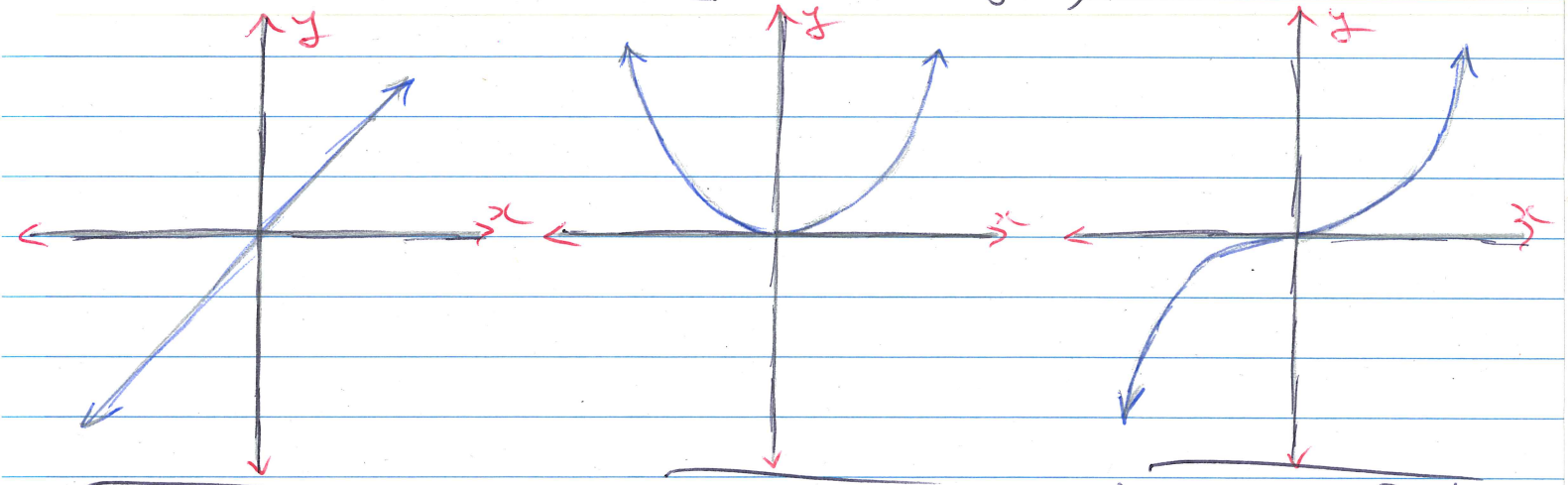
Curve Sketching

Often it proves useful to produce a sketch of a given function. Here we revise some well known functions & then a more systematic method by which any function may be sketched.

Some (hopefully) well known functions

1. Powers of x (positive integers)



$y = x$

$y = x^2$

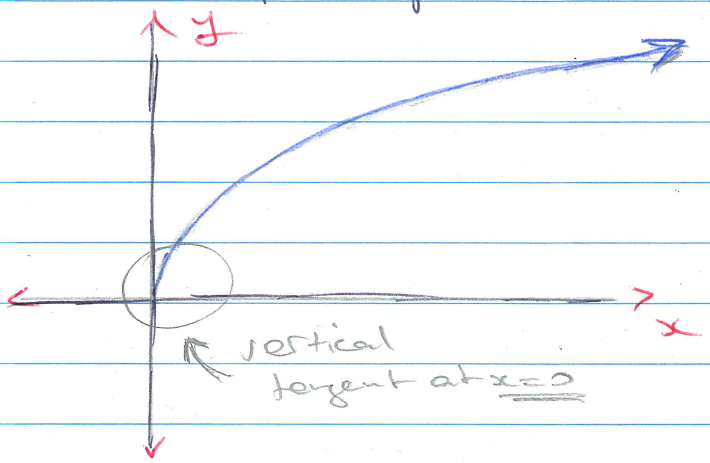
$y = x^4, x^6$, etc look similar to this.

$y = x^3$

$y = x^5, x^7$, etc look similar to this.

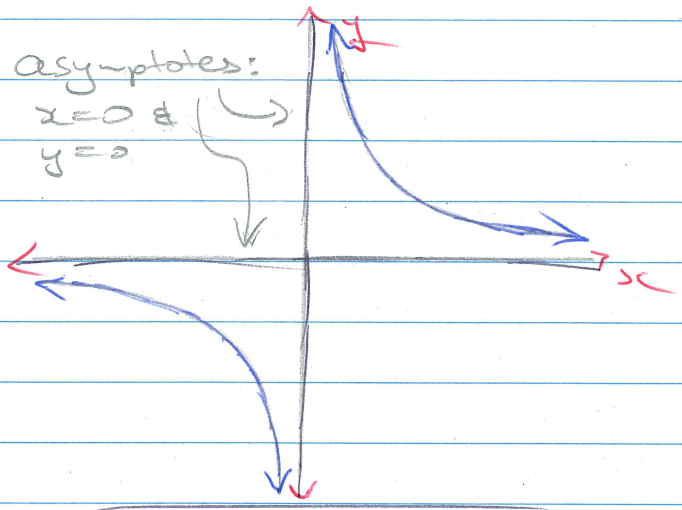
Domain: $x \in \mathbb{R}$

2. Other powers of x.



$$y = \sqrt{x} = x^{\frac{1}{2}}$$

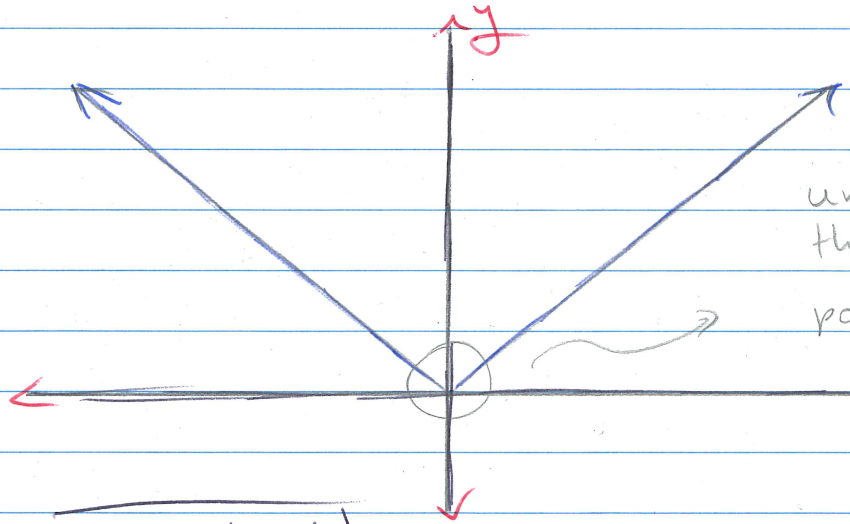
Domain: $x \geq 0$



$$y = \frac{1}{x} = x^{-1}$$

Domain: $x \neq 0$.

3. Absolute value function



unlike the parabola ($y=x^2$) this curve has a sharp point / corner at (0,0)

✓ not ∪

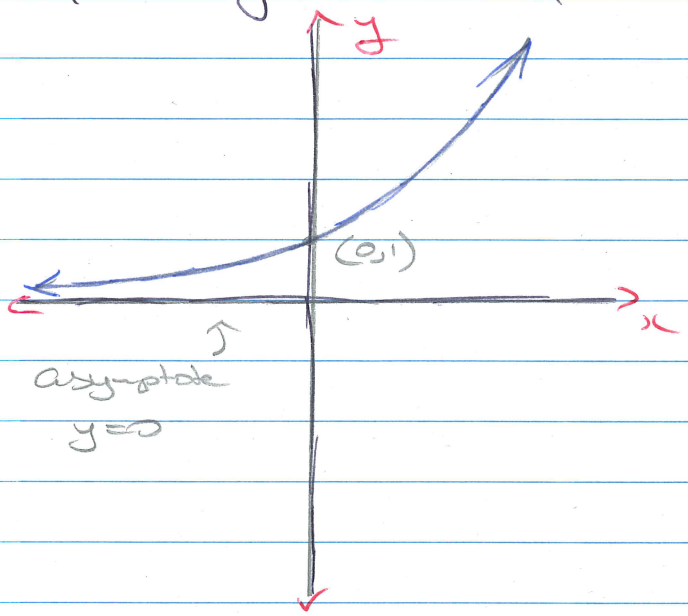
$$y = |x|$$

Domain: $x \in \mathbb{R}$

recall that

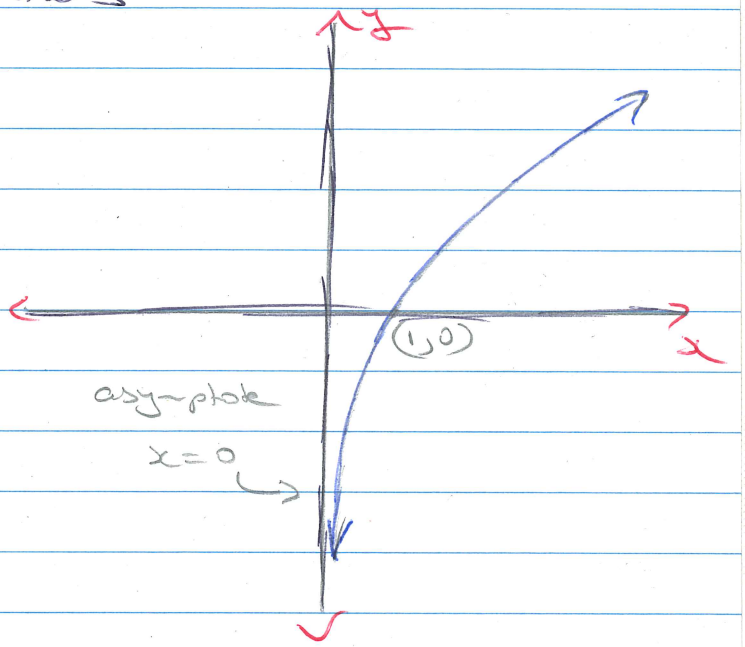
$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0. \end{cases}$$

4. Logarithms & exponentials.



$$y = e^x$$

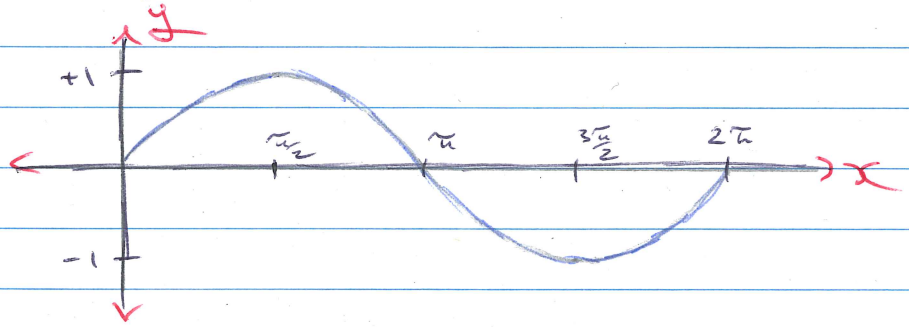
Domain: $x \in \mathbb{R}$



$$y = \ln(x)$$

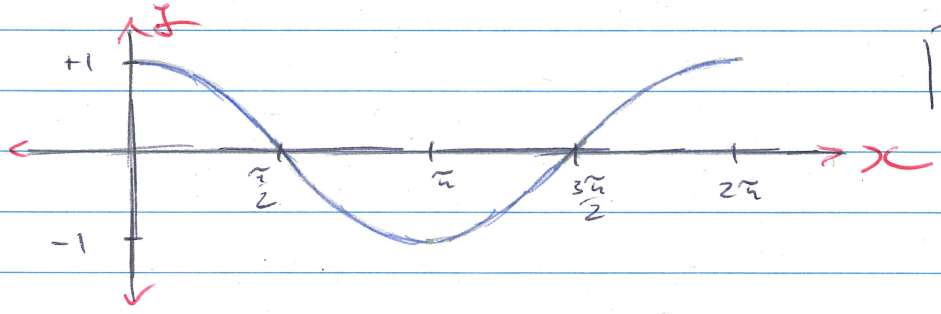
Domain: $x > 0$

5. Trigonometric functions.



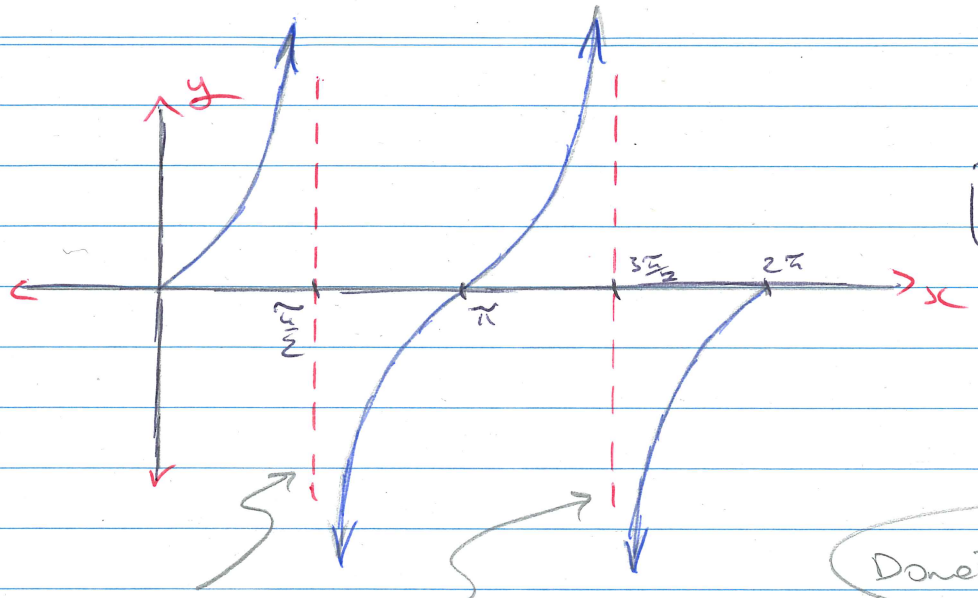
$$y = \sin(x)$$

Domain: $x \in \mathbb{R}$



$$y = \cos(x)$$

Domain: $x \in \mathbb{R}$



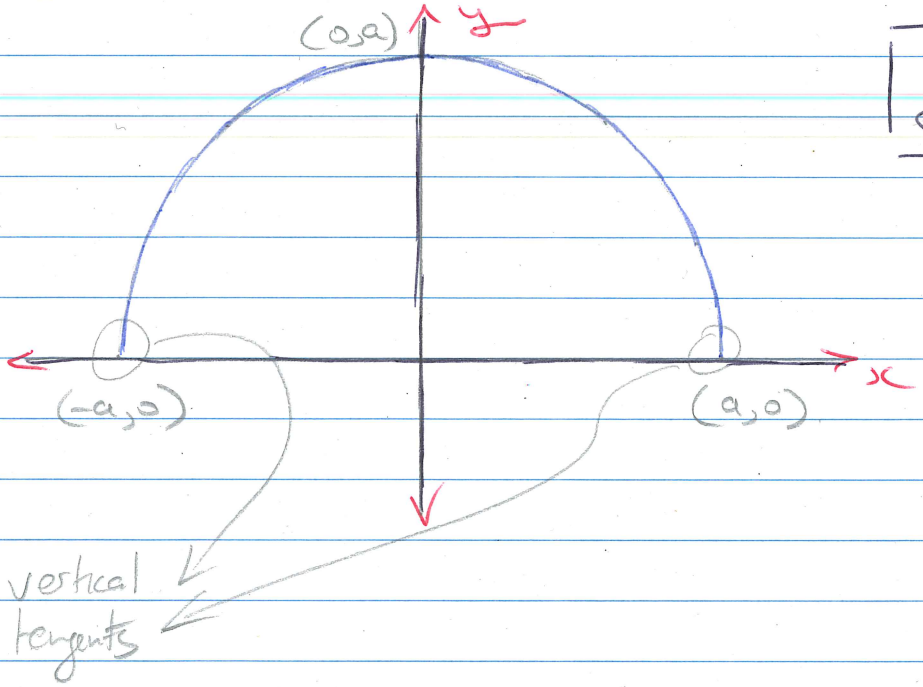
$y = \tan(x)$

Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

asymptotes

$x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

6. Semi-circle



$y = \sqrt{a^2 - x^2}$

Domain: $-a \leq x \leq a$

The following is a list of properties which allow for a comprehensive understanding of a given function under consideration & hence accurate graphing of the function.

(1) Domain & Range

(i) Domain (x-values)
(ii) Range if easy enough (y-values)

(2) Symmetry Properties

Determine if $f(x)$ is even, odd or neither.

(i) if $f(-x) = f(x)$ then $f(x)$ is even.
this implies that $y=f(x)$ is symmetrical about the y-axis.

(ii) if $f(-x) = -f(x)$ then $f(x)$ is odd.
this implies that $y=f(x)$ is symmetrical in the origin.

(iii) if $f(x)$ is neither even nor odd then $y=f(x)$ possesses no obvious symmetry.

(3) Axes Intercepts

(i) x-intercepts:
set $y=0$ & solve for x

(ii) y-intercepts:
set $x=0$ & solve for y.

(4) Vertical Asymptotes

If $f(x)$ is a fraction then there exists vertical asymptotes for x-values at which the denominator is zero.

if $f(x) = \frac{N}{D}$ $\rightsquigarrow$ $D=0$ & solve for x.

(5) Sign of $y=f(x)$

Construct a sign diagram of y. Be sure to include x-values of x-intercepts ($y=0$), vertical asymptotes (y = undefined) and any other important features.

x			
y			

(6) Gradient & Stationary Points

Find $\frac{dy}{dx} = f'(x)$.

(i) Find critical points. (x, y)
- (a) STATIONARY POINTS ($y' = 0$)
- (b) OTHER CRITICAL POINTS ($y' =$ undefined)

(ii) Determine slope along curve.
(y' sign diagram!)

(iii) Characterise stationary points.

(7) Concavity & Points of Inflection.

Find $\frac{d^2y}{dx^2} = f''(x)$.

(i) Find possible points of inflection ($y'' = 0$). (x, y)

(ii) Determine concavity along curve.
(y'' sign diagram!).

(iii) Characterise possible inflection points.

Change in concavity either side of possible inflection point

$\Rightarrow$ Actual Point of INFLECTION

(8) Piece It Together

Plot all known information & check to see if any more work is required.

Usually the only thing left will be to consider behaviour as $x \to \pm\infty$ or similar. The following limits may be useful.

$$\lim_{x \to +\infty} f(x) \quad \& \quad \lim_{x \to -\infty} f(x)$$

Additionally, polynomial long division can assist in plotting functions which are quotients of polynomials by allowing us to identify asymptotes (oblique).

Symmetry & Even and Odd Functions

Even Functions:

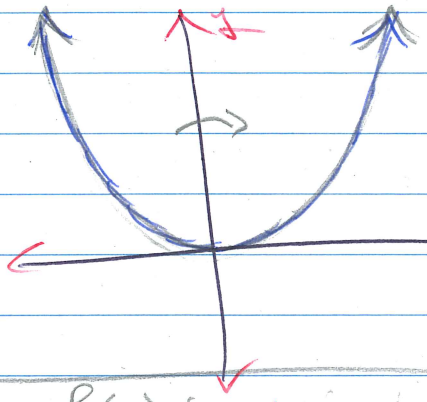
Take for example $f(x) = x^2$

OTHERS

$\cos(x)$,
$|x|$,
$\frac{1}{x^2+1}$

$$f(-x) = (-x)^2$$
$$= x^2$$
$$= f(x).$$

$\therefore$ $f(x)$ is even!



$y = f(x)$ is preserved if reflected in y-axis

Odd Functions:

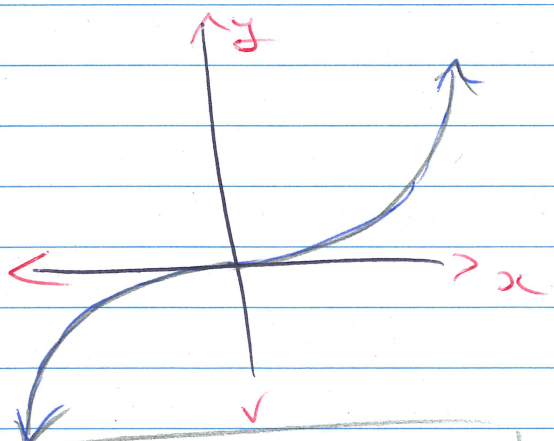
Take for exaple $f(x) = x^3$.

OTHERS

$\sin(x)$,
$\tan(x)$,
$\frac{x}{x^2+1}$

$$f(-x) = (-x)^3$$
$$= -x^3$$
$$= -f(x).$$

$\therefore$ $f(x)$ is odd!



$y = f(x)$ is preserved if rotated about origin 180°.

Sketch $y = \frac{x^3}{x^2-1}$.

① Domain & Range

Domain: $x^2 - 1 \neq 0$

$\boxed{x \neq \pm 1}$

Range: $y \in \mathbb{R}$.

② Symmetry Properties.

$f(x) = \frac{x^3}{x^2-1}$.

$f(-x) = \frac{(-x)^3}{(-x)^2-1}$

$= \frac{-x^3}{x^2-1}$

$= -f(x)$.

$\therefore f(x)$ is odd. ($y = f(x)$ is preserved if rotated 180° about the origin)

(3) Axes Intercepts

$$y = \frac{x^3}{x^2-1}$$

x-int: $(y=0)$

$$0 = \frac{x^3}{x^2-1}$$

$$0 = x^3$$

$$\boxed{x = 0} \quad (0,0)$$

y-int: $(x=0)$

$$y = \frac{0^3}{0^2-1}$$

$$\boxed{y = 0} \quad (0,0)$$

same point!

(4) Vertical Asymptotes

$f(x)$ is a fraction

$\therefore$ Denominator $= 0$ gives VAs.

ie $x^2 - 1 = 0$

$\therefore x = \pm 1$ are both vertical asymptotes.

(5) Sign of $f(x)$

Recall $f(x) = 0$ at $x = 0$ (intercepts)

and $f(x)$ = undefined at $x = \pm 1$ (vertical asymptotes)

x		-1		0		$+1$	
$f(x)$	$-$	undefined	$+$	0	$-$	undefined	$+$

test points for sign of $f(x)$. $y = \frac{x^3}{x^2-1}$

$x = -2$	$x = -\frac{1}{2}$	$x = \frac{1}{2}$	$x = 2$
y = -ve	y = +ve	y = -ve	y = +ve

(6) Gradient & Stationary Points.

$y = \frac{x^3}{x^2-1}$ use quotient rule to derive.

let $u = x^3$ $\quad v = x^2 - 1$

$u' = 3x^2$ $\quad v' = 2x$.

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{3x^2 \times (x^2-1) - 2x(x^3)}{(x^2-1)^2}$$

$$= \frac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2}$$

$$= \frac{x^4 - 3x^2}{(x^2-1)^2}$$

CRITICAL POINTS: ($y'=0$ or y' = undefined)

$y'=0$ if $\frac{x^4-3x^2}{(x^2-1)^2} = 0$.

$$x^4 - 3x^2 = 0.$$

$$x^2(x^2-3) = 0.$$

ie $x^2 = 0$

or $x^2 - 3 = 0$.

sub into $y = \frac{x^3}{x^2-1}$ to find y

$\therefore$

$x = 0$	$y = 0$
$x = +\sqrt{3}$	$y = \frac{3}{2}\sqrt{3}$
$x = -\sqrt{3}$	$y = -\frac{3}{2}\sqrt{3}$

y' = undefined if $(x^2-1)^2 = 0$.

ie $\boxed{x = \pm 1.}$

So $f'(x) = 0$ at $x = 0, \pm\sqrt{3}$ (stationary pts)

& $f'(x)$ = undefined at $x = \pm 1$ (vert. asymptotes).

x	$-\sqrt{3}$		-1		0		$+1$		$+\sqrt{3}$	
$f'(x)$	+	0	−	∄	−	0	−	∄	−	0	+
	↗		↘		↘		↘		↗	

test points for sign of $f'(x)$.

$$y' = \frac{x^4 - 3x^2}{(x^2-1)^2} = \frac{x^2(x^2-3)}{(x^2-1)^2}$$

$x = -2$ $\quad y' = +ve$

$x = -1.5$ $\quad y' = -ve$

$x = -0.5$ $\quad y' = -ve$

$x = +0.5$ $\quad y' = -ve$

$x = +1.5$ $\quad y' = -ve$

$x = +2$ $\quad y' = +ve$

therefore A $\left(-\sqrt{3}, -\frac{3}{2}\sqrt{3}\right)$ is a local max

B $(0, 0)$ is a horizontal P of I

C $\left(\sqrt{3}, \frac{3}{2}\sqrt{3}\right)$ is a local min.

P of I = Point of Inflection

$\sqrt{3} \approx 1.7$

(7) Concavity & Points of Inflection.

if $\frac{dy}{dx} = \frac{x^4-3x^2}{(x^2-1)^2}$ use the quotient rule to derive

let $u = x^4-3x^2$ $\quad v = (x^2-1)^2$

$u' = 4x^3-6x$ $\quad v' = 2(x^2-1)\times 2x$

↑ use chain rule for this.

$$\therefore \frac{d^2y}{dx^2} = \frac{u'v - v'u}{v^2}$$

$$= \frac{(4x^3-6x)(x^2-1)^2 - 4x(x^2-1)(x^4-3x^2)}{(x^2-1)^4}$$

$$= \frac{(4x^3-6x)(x^2-1) - 4x(x^4-3x^2)}{(x^2-1)^3}$$

$$= \frac{\cancel{4x^5} - 4x^3 - 6x^3 + 6x \cancel{- 4x^5} + 12x^3}{(x^2-1)^3}$$

$$= \frac{2x^3+6x}{(x^2-1)^3}$$

Possible points of Inflection

$y'' = 0$. if $\dfrac{2x^3+6x}{(x^2-1)^3} = 0$.

$2x^3 + 6x = 0$.

$2x(x^2+3) = 0$.

$\boxed{x = 0}$.

$x^2 + 3 \neq 0$

x		-1		0		$+1$	
$f''(x)$	$-$	‖	$+$	0	$-$	‖	$+$
	∩		∪	·	∩		∪

⇒ CHANGE IN CONCAVITY.

test points for sign of $f''(x)$ $\quad y'' = \dfrac{2x^3+6x}{(x^2-1)^3}$

$x = -2$ $\quad y'' = -ve$

$x = -\frac{1}{2}$ $\quad y'' = +ve$ $\quad \rightarrow$

$x = +\frac{1}{2}$ $\quad y'' = -ve$

$x = +2$. $\quad y'' = +ve$

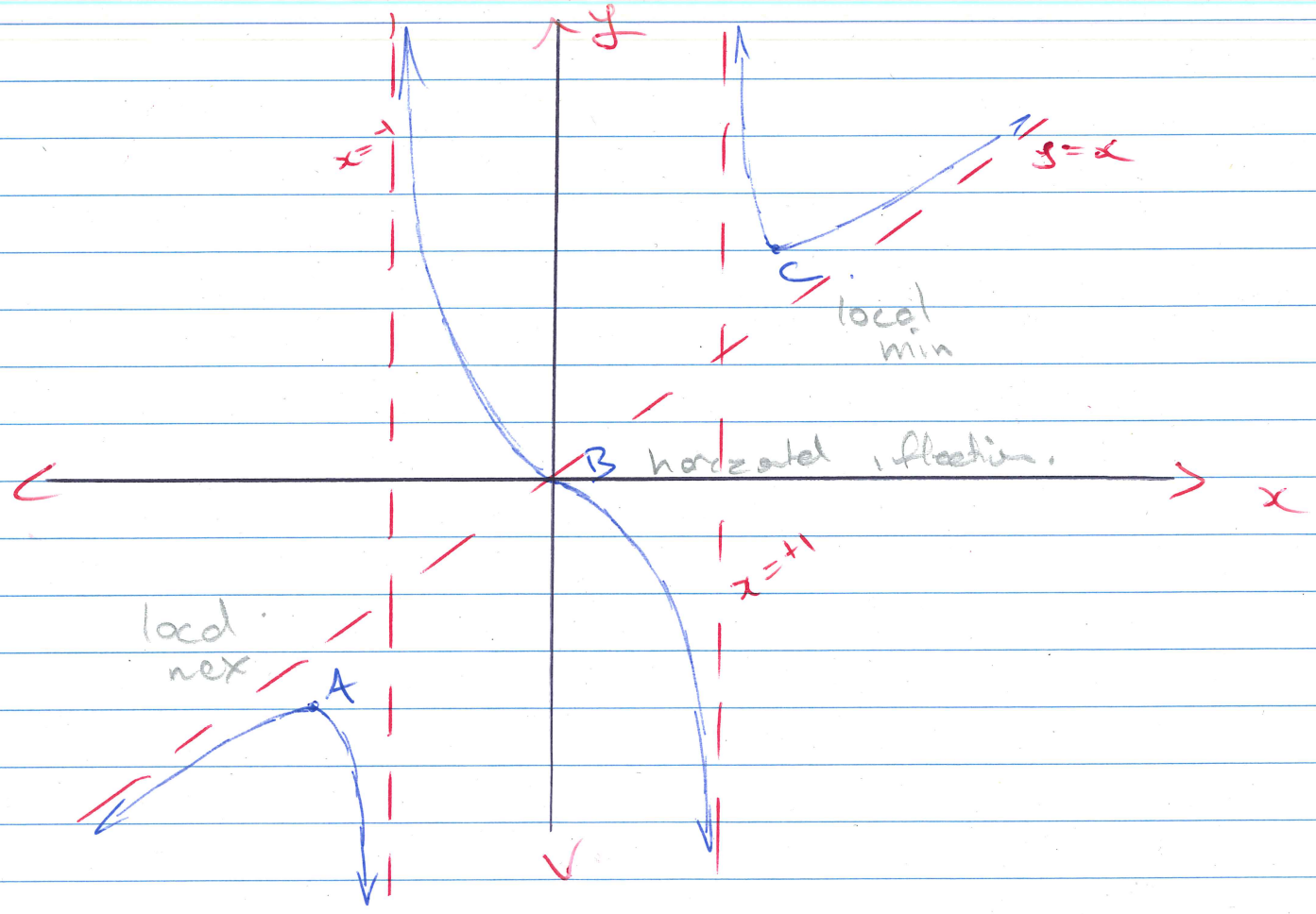
$= \dfrac{2x(x^2+3)}{(x^2-1)^3}$

the point B(0,0) is a point of inflection.

(8) Piece It Together.

$$\lim_{x\to\infty} \frac{x^3}{x^2-1} \sim x.$$

$$\lim_{x\to-\infty} \frac{x^3}{x^2-1} \sim x.$$

- if x is big then x^2 is huge.
- x^2-1 will look very much like x^2.
- so $\frac{x^3}{x^2-1}$ looks like $\frac{x^3}{x^2} = x$ for large x

$\boxed{y=x}$ is an oblique asymptote