Integral Calculus

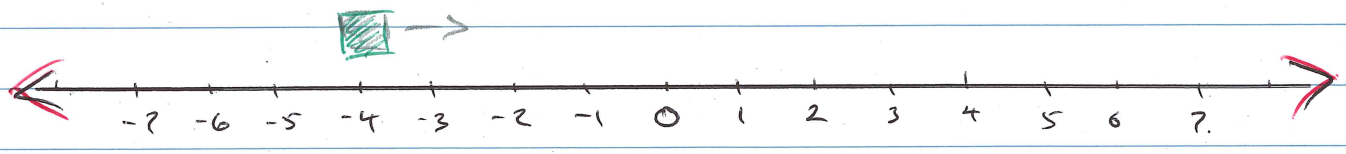
Calculus can be thought of as possessing two branches; Differential calculus & Integral calculus.

While differential calculus deals with the derivative of a function, how to obtain the derivative and what it means, integral calculus deals with the integral of a function, how to obtain it and what it means.

We note here, and shall show later, that the process of integration is the reverse process to differentiation.

To introduce the concepts of integral calculus we consider the relationship between calculus and particle motion.

Consider a particle moving along the number line



1. DISPLACEMENT

The displacement of the particle gives the position of the particle along the number line. (Not distance from 0!)

x - displacement of particle from origin.

Distance → always positive

Displacement → can be negative & denotes being to the left of the origin

2. VELOCITY

The velocity of a particle gives a directed speed along the number line. (Not just speed!)

v - velocity of particle.

Speed → always positive

velocity → can be negative & denotes moving to the left.

Note: velocity is the rate of change of displacement with respect to time.

ie $v = \frac{dx}{dt}$.

if $x = f(t)$

$v = \frac{dx}{dt} = f'(t)$

3. ACCELERATION

The acceleration of a particle tells us how the velocity changes.

a - acceleration of particle.

if POSITIVE, v is increasing.
if NEGATIVE, v is decreasing.

Note; Acceleration is the rate of change of velocity with respect to time.

ie. $a = \frac{dv}{dt}$

if $v = f(t)$

$a = \frac{dv}{dt} = f'(t)$

and so,

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

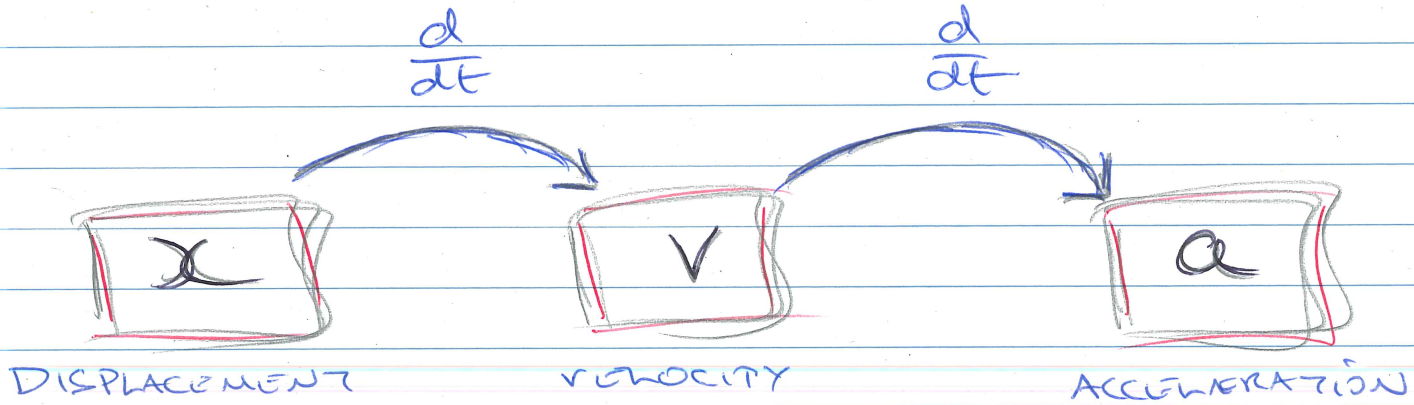
if $x = f(t)$

$v = \frac{dx}{dt} = f'(t)$

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = f''(t)$.

The complete motion of a particle is adequately described by the three variables x, v & a.

- If we are given $x = f(t)$, using differential calculus ...



we can move from x to v and from v to a.

BUT WHAT ABOUT MOVING IN THE OTHER DIRECTION?

Looks like a process which "undoes" differentiation! $\left(v \rightarrow x \;\&\; a \rightarrow v\right)$

Velocity to Displacement:

Ex 1: Consider a particle moving to the right at 4 m/s, ie, $v = +4$ m/s. Find the displacement of the particle at any time t.

- Each second the displacement changes by 4 meters
  $\Rightarrow$ In t seconds the displacement changes by $4t$ meters.

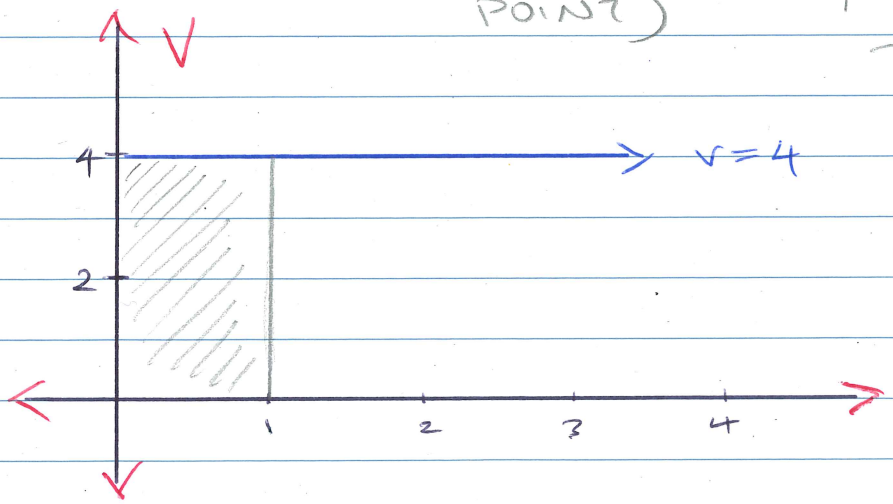
- Displacement depends on the initial position of the particle.

$$x = x_0 + 4t$$

if $x = x_0 + 4t$, $\frac{dx}{dt} = 4$

$x = x_0$ if $t = 0$ (START POINT)

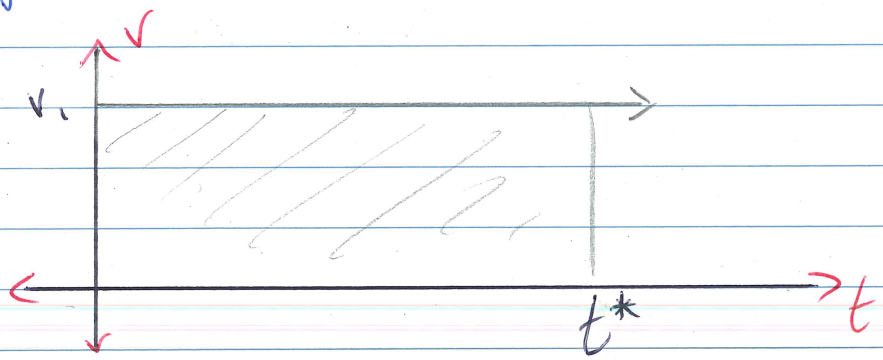
CHANGE IN x from $v = \frac{dx}{dt}$



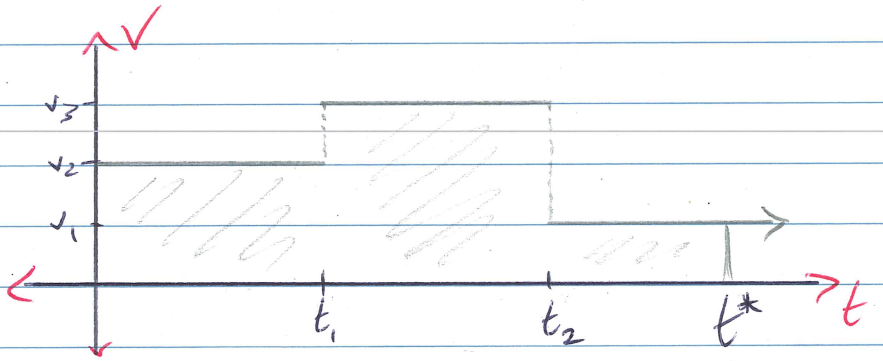
area below curve: gives change in x !!!

Here we see that the process of moving backwards through differentiation is linked to finding areas below a curve, more specifically we could construct x by considering areas below the curve $v = v(t)$.

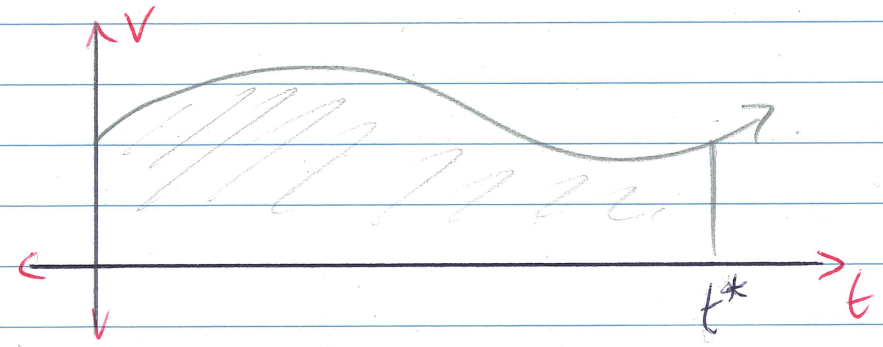
However this was a simple example, let us consider a more general process.



$$\text{Area} = t^* \times v_1$$



$$\text{Area} = (t_1 - 0) \times v_2 + (t_2 - t_1) \times v_3 + (t^* - t_2) \times v_1$$

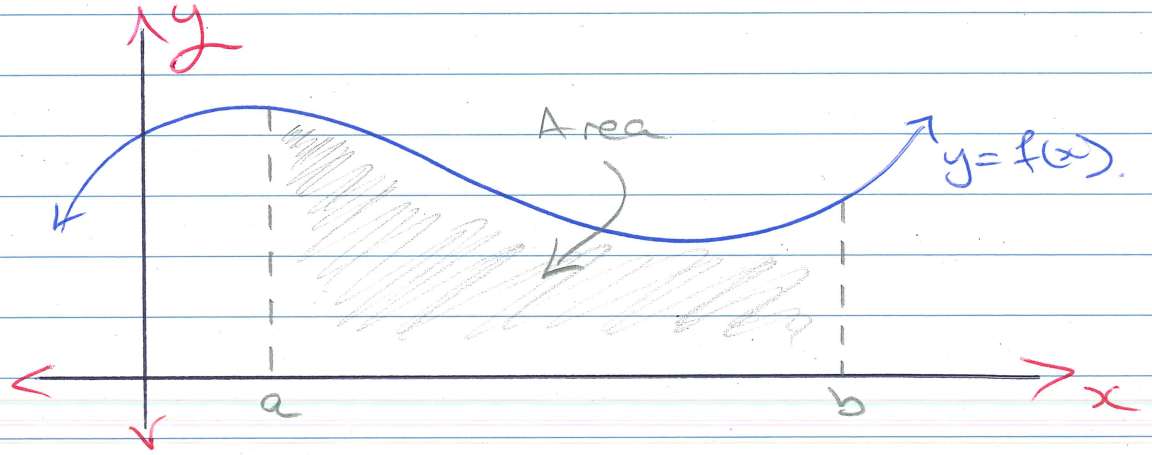


$$\text{Area} = ?$$

Riemann Sums:

The process of approximating an area under a curve using rectangles is known as calculating Riemann sums.

Consider a function $f(x)$ on the interval $[a, b]$.



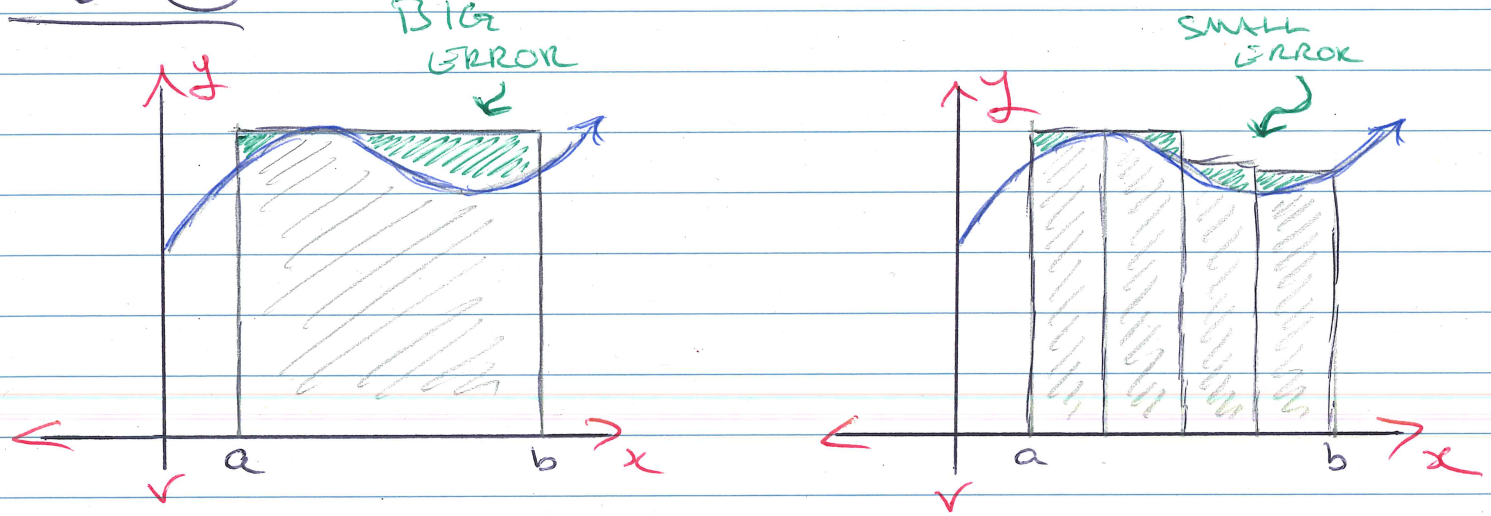
The area between this curve and the x-axis may be approximated by considering appropriate rectangles.

- In general a Riemann sum will not give an exact area under a curve but rather an approximation.

Riemann Sum Calculations

For any given area we consider two types of Riemann sums; an upper Riemann sum & a lower Riemann sum. Each of these can be calculated in two steps.

STEP (1)



Divide the interval $[a,b]$ into sub-intervals.

$$\#\text{ of sub-intervals} = n$$

$$\text{width of sub-intervals} = \frac{b-a}{n} = \Delta x.$$

$x_0 = a$

$x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$

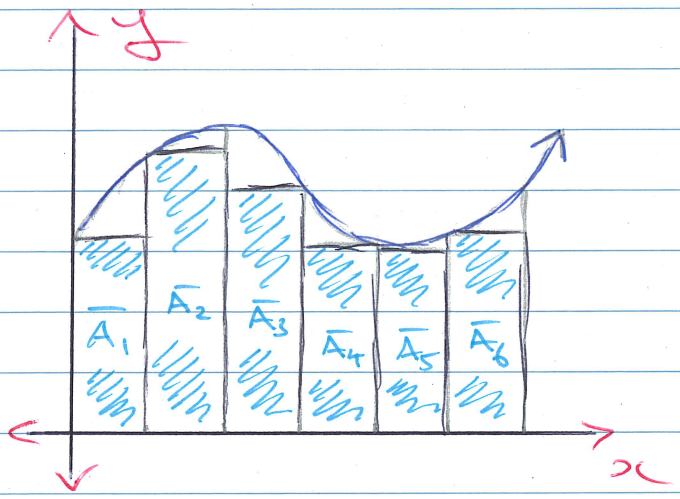
$\vdots$

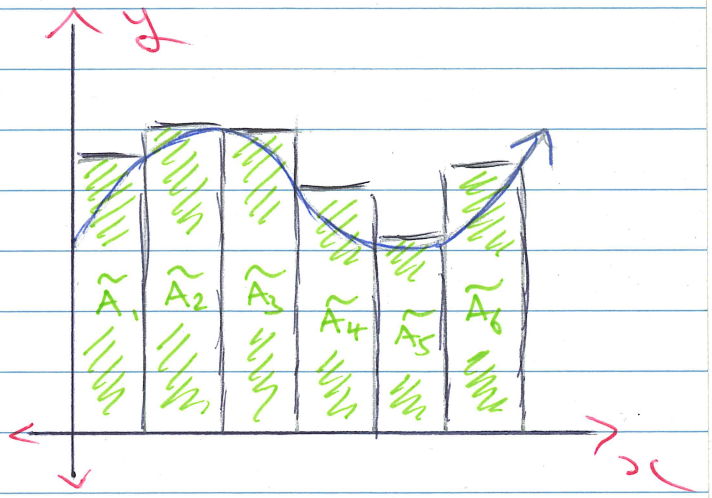
$x_n = b$

$[a,b]$ becomes

$[x_0, x_1] \cup [x_1, x_2] \cup [x_2, x_3] \cup \ldots \cup [x_{n-1}, x_n]$.

STEP ②





Lower R. sum

$$L = \bar{A}_1 + \bar{A}_2 + \bar{A}_3 + \bar{A}_4 + \bar{A}_5 + \bar{A}_6$$

Choose lowest y-value in each interval as height of rectangle.

Area $= y^* \times \Delta x$.

Upper R. sum

$$U = \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3 + \tilde{A}_4 + \tilde{A}_5 + \tilde{A}_6$$

Choose highest y-value in each interval as height of rectangle.

Area $= y^* \times \Delta x$.

For each interval found in step ①, determine the height of the approximating rectangle. To do this choose the upper or lower of the edges.

on $[x_2, x_3]$ { pick max of $f(x_2)$ & $f(x_3) \to U$; pick min of $f(x_2)$ & $f(x_3) \to L$