The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1	**First Assignment**	2019

This assignment comprises two questions and is worth 5% of the overall assessment. It should be completed, scanned and uploaded using Turnitin through the MATH2022 Canvas portal by 11:59 pm on Thursday 11 April. **Please do not include your name, as anonymous marking will be implemented.**

The first question explores two similar but contrasting arithmetics with four elements, involving 2×2 *matrices with entries from* $\mathbb{Z}_2$*, one of which turns out to be a field.*

The second question involves calculating the determinant and inverting a particular 4×4 *matrix over* $\mathbb{R}$*, and exploring consequences when the elements are then regarded as coming from* $\mathbb{Z}_n$ *for various* n*.*

Your tutor will give you feedback and allocate an overall letter grade (and mark) using the following criteria:

- *A^+ (10): excellent and scholarly work, answering all parts of both questions, with clear and accurate explanations and working, with appropriate acknowledgement of sources, if appropriate, and at most minor or trivial errors or omissions;*
- *A(9): excellent work, making progress on both questions, but with one or two substantial omissions, errors or misunderstandings overall;*
- *B^+ (8): very good work, making progress on both questions, but with three or four substantial omissions, errors or misunderstandings overall;*
- *B(7): good work, making substantial progress on both questions, but making five or six substantial omissions, errors or misunderstandings overall;*
- *C^+ (6): reasonable attempt, making substantial progress on both questions, but making seven or eight substantial omissions, errors or misunderstandings overall;*
- *C(5): reasonable attempt, making progress on both questions, but making more than eight substantial omissions, errors or misunderstandings overall;*
- *D(4): making progress on just one question;*
- *E(2): some attempt, but making no real progress on either question;*
- *F(0): no real attempt at either question.*

1. Consider the following matrices with entries from $\mathbb{Z}_2 = \{0,1\}$:

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Consider the sets

$$\mathcal{F} = \{O, I, A, B\} \qquad \text{and} \qquad \mathcal{G} = \{O, I, C, D\},$$

both of which form arithmetics with respect to matrix addition and multiplication.

(a) Complete the following addition tables for $\mathcal{F}$ and $\mathcal{G}$:

$+$	O	I	A	B
O				
I				
A				
B				

$+$	O	I	C	D
O				
I				
C				
D				

(b) Complete the following multiplication tables for $\mathcal{F}$ and $\mathcal{G}$:

$\cdot$	O	I	A	B
O				
I				
A				
B				

$\cdot$	O	I	C	D
O				
I				
C				
D				

(c) Explain why $\mathcal{G}$ forms a field with respect to addition and multiplication. (You may quote relevant properties of matrix addition and multiplication without proof.)

(d) Give a brief reason why the arithmetic $\mathcal{F}$ does not form a field.

2. Consider the following 4×4 matrix:

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

(a) Find $\det M$, the determinant of M, working over $\mathbb{R}$.

(b) Use part (a) to deduce the value of $\det M$, working over each of $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_7$.

(c) Find M^{-1}, the matrix inverse of M, working over $\mathbb{R}$.

(d) Use part (c) to deduce M^{-1}, working over each of $\mathbb{Z}_2$ and $\mathbb{Z}_7$. Explain briefly how you know M^{-1} does not exist when working over $\mathbb{Z}_3$.

(e) Solve the following system of equations, working over each of $\mathbb{R}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_7$:

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & & & = & 0 \\ x_1 & + & x_2 & + & & & x_4 & = & 1 \\ x_1 & + & & & x_3 & + & x_4 & = & 1 \\ & & x_2 & + & x_3 & + & x_4 & = & 0 \end{array}$$