The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 — **First Quiz Practice Exercises** — 2019

The First Quiz on 25 March 2019 lasts forty minutes and will consist of fifteen multiple choice exercises, similar to a selection from the exercises below.

1. Which one of the following forms a field under addition and multiplication?

(a) $\mathbb{N}$ (b) $\mathbb{Z}_2$ (c) $\mathbb{Z}_4$ (d) $\mathbb{Z}_6$ (e) $\mathbb{Z}_8$

2. Which one of the following is not a field?

(a) $\mathbb{Q}$ (b) $\mathbb{R}$ (c) $\mathbb{C}$ (d) $\mathbb{Z}$ (e) $\mathbb{Z}_{13}$

3. If today is Thursday, what day of the week will it be after 2018^{2018} days have elapsed?

(a) Friday (b) Saturday (c) Sunday
(d) Monday (e) Tuesday

4. Which one of the following statements is true?

(a) $\frac{2}{3} = 5$ in $\mathbb{Z}_{11}$. (b) $\frac{3}{4} = 4$ in $\mathbb{Z}_{13}$. (c) $\frac{3}{4} = 2$ in $\mathbb{Z}_7$.
(d) $\frac{2}{3} = 6$ in $\mathbb{Z}_{13}$. (e) $\frac{3}{4} = 7$ in $\mathbb{Z}_{11}$.

5. Consider the following matrix

$$M = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 4 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

with entries from $\mathbb{Z}_7$. Working over $\mathbb{Z}_7$, which of the following is true?

(a) $\det M = 0$ (b) $\det M = 4$ (c) $\det M = 5$
(d) $\det M = 2$ (e) $\det M = 6$

6. Consider the following system of equations over $\mathbb{Z}_5$:

$$\begin{array}{rcrcrcrcl} x & + & 2y & & & + & w & = & 1 \\ 2x & + & y & + & z & & & = & 2 \\ x & + & y & + & 2z & + & 2w & = & 1 \end{array}$$

Working over $\mathbb{Z}_5$, how many distinct solutions are there for (x, y, z, w)?

(a) infinitely many (b) no solutions (c) exactly one
(d) exactly five (e) exactly twenty-five

7. Find the unique solution to the following matrix equation

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

working over $\mathbb{Z}_2$.

(a) $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

8. Find the value of λ such that the system

$$\begin{array}{rcrcrcr} x & & & + & 2z & = & 2 \\ -x & + & \lambda y & + & z & = & -1 \\ x & - & y & - & \lambda z & = & 1 \end{array}$$

is inconsistent over $\mathbb{Z}_5$, but has a unique solution over $\mathbb{R}$.

(a) $\lambda = 0$ (b) $\lambda = 1$ (c) $\lambda = 2$

(d) $\lambda = 3$ (e) $\lambda = 4$

9. Consider the matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

with entries from $\mathbb{Z}_2$. Which of the following is row equivalent to M and in reduced row echelon form?

(a) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

10. Consider the following matrices over $\mathbb{R}$, where θ is a real number:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad T_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Which one of the following statements is true?

(a) $R^3_{\pi/3} = I = T^2_{\pi/2}$ (b) $R^3_{2\pi/3} = I = T^3_{2\pi/3}$ (c) $R^4_{\pi/4} = I = T^4_{\pi/4}$
(d) $R_{\pi/2}T_{2\pi/3}R_{\pi/2} = T_{4\pi/3}$ (e) $T_{\pi/2}R_{2\pi/3}T_{\pi/2} = R_{4\pi/3}$

11. Consider the real matrix

$$M = \begin{bmatrix} 0 & 2 \\ 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 3 & 9 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and elementary matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Use the chain of equivalences above, or otherwise, to find a correct expression for M as a product of these elementary matrices.

(a) $M = E_4E_2E_3E_1$ (b) $M = E_4E_2E_1E_3$ (c) $M = E_3E_2E_1E_4$
(d) $M = E_4E_3E_2E_1$ (e) $M = E_2E_1E_3E_4$

12. Suppose that A, B and P are real square matrices such that P is invertible and $\lambda \in \mathbb{R}$ such that

$$P(A - \lambda I)P^{-1} = B,$$

where I denotes the identity matrix. Which of the of the following is a correct expression for A?

(a) $A = P^{-1}BP + \lambda I$ (b) $A = PBP^{-1} + \lambda I$ (c) $A = P^{-1}BP + \lambda P$
(d) $A = P^{-1}B + \lambda P^{-1}$ (e) $A = P^{-1}BP + \lambda P^{-1}$

13. Suppose that a, b, c, d, g are elements of a group G such that

$$abgc^{-1} = d.$$

Which one of the following is a correct expression for g?

(a) $g = a^{-1}b^{-1}dc$ (b) $g = dca^{-1}b^{-1}$ (c) $g = (ab)^{-1}cd$
(d) $g = b^{-1}a^{-1}dc$ (e) $g = c(ab)^{-1}d$

14. Consider the permutations

$$\alpha = (1\,2\,3\,4)(5\,6\,7), \qquad \beta = (1\,3)(2\,4), \qquad \gamma = (1\,2\,3)(4\,5)(6\,7)$$

of $\{1, 2, 3, 4, 5, 6, 7\}$ expressed in cycle notation. Which one of the following is correct?

(a) β and γ are even, and α is odd. (b) α and β are even, and γ is odd.
(c) α and γ are even, and β is odd. (d) α and β are odd, and γ is even.
(e) α and γ are odd, and β is even.

15. Consider the permutation

$$\alpha \;=\; (1\;2\;3)(6\;3\;2)(5\;4\;6\;3\;2)$$

of $\{1,2,3,4,5,6\}$ expressed in cycle notation where we compose from left to right. Which one of the following is a correct equivalent expression?

(a) $\alpha = (1\;2\;3)(5\;6\;4)$ (b) $\alpha = (1\;3)(2\;5\;4\;6)$ (c) $\alpha = (1\;3)(2\;6\;4\;5)$
(d) $\alpha = (1\;4)(3\;6\;5\;2)$ (e) $\alpha = (1\;4\;2)(3\;5\;6)$

16. Consider the permutations

$$\alpha \;=\; (1\;2\;3\;4)(5\;6\;7)\,, \qquad \beta \;=\; (1\;3)(2\;4)\,, \qquad \gamma \;=\; (1\;2\;3)(4\;5)(6\;7)$$

of $\{1,2,3,4,5,6,7\}$ expressed in cycle notation. Simplify the permutation

$$\delta \;=\; \alpha\beta\gamma^{-1}\,,$$

composing from left to right:

(a) $\delta = (1\;5\;7\;4\;2\;3)$ (e) $\delta = (1\;5\;7\;4)$ (d) $\delta = (1\;3\;2\;4\;7\;5)$
(c) $\delta = (1\;4\;7\;5)$ (b) $\delta = (1\;5\;7\;4)(3\;2)$

17. Consider the permutations

$$\alpha \;=\; (1\;3)(2\;4\;6\;5) \qquad \text{and} \qquad \beta \;=\; (1\;4\;2\;5)(6\;3)$$

of $\{1,2,3,4,5,6\}$ expressed in cycle notation. Which one of the following is a correct expression for the permutation

$$\gamma = \beta^{-1}\alpha\beta$$

where we compose from left to right?

(a) $\gamma = (4\;6)(5\;2\;1\;3)$ (b) $\gamma = (5\;6)(4\;3\;1\;2)$ (c) $\gamma = (4\;6)(1\;5\;2\;3)$
(d) $\gamma = (4\;6)(5\;3\;1\;2)$ (e) $\gamma = (5\;6)(4\;1\;3\;2)$

18. Consider the permutations

$$\alpha \;=\; (1\;3)(4\;2\;5\;6) \qquad \text{and} \qquad \gamma \;=\; (4\;5)(1\;3\;2\;6)$$

of $\{1,2,3,4,5,6\}$ expressed in cycle notation. Which one of the following is a correct expression for a permutation β with the property

$$\gamma \;=\; \beta^{-1}\alpha\beta$$

where we compose from left to right?

(a) $\beta = (1\;6)(2\;3)$ (b) $\beta = (1\;4\;2\;3\;6\;5)$ (c) $\beta = (1\;4\;6)(2\;3\;5)$
(d) $\beta = (1\;4)(2\;6)(3\;5)$ (e) $\beta = (1\;4\;2\;6\;3\;5)$

19. Which one of the following configurations is possible to reach from the 8-puzzle

1	2	3
4	5	6
7	8	

by moving squares in and out of the space?

(a)

4	1	5
6	2	3
7	8	

(b)

7	6	4
2	8	3
1	5	

(c)

8	6	4
3	1	5
2	7	

(d)

2	3	4
8	7	5
1	6	

(e)

4	8	2
5	7	1
3	6	

20. Which one of the following configurations is impossible to reach from the 8-puzzle

1	2	3
4	5	6
7	8	

by moving squares in and out of the space?

(a)

2	4	
8	1	5
3	6	7

(b)

7	2	4
6		8
1	5	3

(c)

4	8	2
3	6	
1	7	5

(d)

8	3	4
2		7
6	5	1

(e)

	7	1
6	4	2
3	5	8