The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1	**Second Assignment**	2019

This assignment comprises two questions and is worth 5% *of the overall assessment. It should be completed, scanned and uploaded using Turnitin through the MATH2022 Canvas portal by 11:59 pm on Thursday 16 May.* **Please do not include your name, as anonymous marking will be implemented.**

The first question is an extended question involving diagonalising a matrix, applying the Cayley-Hamilton Theorem to find a matrix inverse, and an application to solving a system of differential equations.

The second question explores an arithmetic involving real 2×2 *matrices, which turns out to be a field and, in fact, an isomorphic copy of* $\mathbb{C}$ *, the field of complex numbers.*

Your tutor will give you feedback and allocate an overall letter grade (and mark) using the following criteria:

A^+(10): *excellent and scholarly work, answering all parts of both questions, with clear and accurate explanations and working, with appropriate acknowledgement of sources, if appropriate, and at most minor or trivial errors or omissions;*

A(9): *excellent work, making progress on both questions, but with one or two substantial omissions, errors or misunderstandings overall;*

B^+(8): *very good work, making progress on both questions, but with three or four substantial omissions, errors or misunderstandings overall;*

B(7): *good work, making substantial progress on both questions, but making five or six substantial omissions, errors or misunderstandings overall;*

C^+(6): *reasonable attempt, making substantial progress on both questions, but making seven or eight substantial omissions, errors or misunderstandings overall;*

C(5): *reasonable attempt, making progress on both questions, but making more than eight substantial omissions, errors or misunderstandings overall;*

D(4): *making progress on just one question;*

E(2): *some attempt, but making no real progress on either question;*

F(0): *no real attempt at any question.*

1. Consider the following real matrix:

$$M = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

(a) Verify that the characteristic polynomial of M is

$$\chi(\lambda) = \lambda^3 - \lambda^2 - 5\lambda - 3 = (\lambda+1)^2(\lambda-3) .$$

(b) Use the Cayley-Hamilton Theorem to deduce that M is invertible and

$$M^{-1} = \frac{1}{3}\left(M^2 - M - 5I\right) .$$

(c) Find the eigenvalues of M and the corresponding eigenspaces.

(d) Find an invertible matrix P and a diagonal matrix D such that $M = PDP^{-1}$.

(e) Consider three differentiable functions

$$x = x(t)\,, \quad y = y(t)\,, \quad z = z(t)$$

of a real variable t linked by the following system of differential equations:

$$\begin{array}{rcrcrcr} x' & = & x & & & + & 2z \\ y' & = & x & - & y & + & z \\ z' & = & 2x & & & + & z \end{array}$$

Find the particular solution to this system given that $x(0) = 1$, $y(0) = 2$ and $z(0) = 3$.

2. Consider the following set of real 2×2 matrices:

$$F = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

(a) Verify that F is closed under matrix addition and taking negatives (so F forms an abelian group with respect to addition).

(b) Verify that F is closed under matrix multiplication and taking matrix inverses of nonzero elements.

(c) Verify that matrix multiplication is commutative when restricted to F. (This, together with usual properties of matrix addition and multiplication, completes the verification that F forms a field).

(d) Let $\phi : \mathbb{C} \to F$ be the map

$$a + bi \;\mapsto\; \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

for $a, b \in \mathbb{R}$, where $i = \sqrt{-1}$. Then clearly ϕ is a bijection (and there is no need to verify this). Verify that

$$(z + w)\phi \;=\; z\phi + w\phi \qquad \text{and} \qquad (zw)\phi \;=\; (z\phi)(w\phi)\;.$$

for all $z, w \in \mathbb{C}$. (This verifies that $\mathbb{C}$ and F are *isomorphic* fields.)