The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 | **Second Quiz Practice Exercises** | 2019

The Second Quiz on 29 April 2019 lasts forty minutes and will consist of fifteen multiple choice exercises, similar to a selection from the exercises below.

1. Consider the following system of equations over $\mathbb{Z}_3$:

$$\begin{array}{rcrcrcrcl} x & + & 2y & & & + & w & = & 1 \\ 2x & + & y & + & z & & & = & 2 \\ x & + & 2y & + & z & + & 2w & = & 1 \end{array}$$

Working over $\mathbb{Z}_3$, how many distinct solutions are there for (x, y, z, w)?

(a) infinitely many (b) no solutions (c) exactly one
(d) exactly three (e) exactly nine

2. Consider the following matrices over $\mathbb{R}$, where θ is a real number:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad , \quad T_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Which one of the following statements is true?

(a) $R^3_{\pi/3} = I = T^2_{\pi/3}$ (b) $R^2_\pi = I = T^3_\pi$ (c) $R^8_{\pi/4} = I = T^8_{\pi/4}$
(d) $R_{\pi/2}T_{2\pi/3}R_{\pi/2} = T_{4\pi/3}$ (e) $T_{\pi/2}R_{2\pi/3}T_{\pi/2} = R_{2\pi/3}$

3. Consider the real matrix

$$M = \begin{bmatrix} 0 & 2 \\ 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 3 & 9 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and elementary matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad E_2 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Use the chain of equivalences above, or otherwise, to express $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as a product of these elementary matrices with M.

(a) $I = ME_4E_2E_1E_3$ (b) $I = E_4E_2ME_1E_3$ (c) $I = E_3E_1E_2E_4M$
(d) $I = E_4E_2E_1E_3M$ (e) $I = ME_3E_1E_2E_4$

4. Working over $\mathbb{Z}_5$, the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ are

(a) 1 and 2. (b) 2 and 3. (c) 4 only.
(d) 2 and 4. (e) 1 only.

5. Which one of the following is a true statement about the real matrix $M = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}$?

(a) -2 is an eigenvalue of M with corresponding eigenspace $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

(b) 2 is an eigenvalue of M with corresponding eigenspace $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

(c) 3 is an eigenvalue of M with corresponding eigenspace $\left\{ \begin{bmatrix} -t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

(d) -2 is an eigenvalue of M with corresponding eigenspace $\left\{ \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

(e) -3 is an eigenvalue of M with corresponding eigenspace $\left\{ \begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

6. Let $M = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$ with entries from $\mathbb{Z}_7$. Then $M = PDP^{-1}$ where

(a) $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} 6 & 1 \\ 4 & 1 \end{bmatrix}$

(b) $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 6 \\ 1 & 4 \end{bmatrix}$

(c) $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 4 & 6 \end{bmatrix}$

(d) $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 6 & 1 \\ 4 & 1 \end{bmatrix}$

(e) $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 6 \\ 1 & 4 \end{bmatrix}$

7. Working over $\mathbb{R}$, suppose that $M = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then, for any positive integer k, we have that M^k is

(a) $\begin{bmatrix} -3^k & 2^k - 3^k \\ 0 & -2^k \end{bmatrix}$

(b) $\begin{bmatrix} 3^k & 2^k - 3^k \\ 0 & 2^k \end{bmatrix}$

(c) $\begin{bmatrix} 2^k & 3^k - 2^k \\ 0 & 3^k \end{bmatrix}$

(d) $\begin{bmatrix} 2^k & 1 \\ 0 & 3^k \end{bmatrix}$

(e) $\begin{bmatrix} 2k & k \\ 0 & 3k \end{bmatrix}$

8. The characteristic polynomial of the real matrix $M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ is

(a) $\lambda^3 - 5\lambda^2 + 6\lambda$.

(b) $\lambda^3 - 5\lambda^2 + 8\lambda - 4$.

(c) $\lambda^3 + 5\lambda^2 + 8\lambda + 4$.

(d) $\lambda^3 - \lambda^2 - 4\lambda + 4$.

(e) $\lambda^3 - 5\lambda^2 + 4\lambda + 4$.

9. Which of the following expressions describes M^{-1} where $M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ and I is the 3×3 identity matrix, working over $\mathbb{R}$?

(a) $\frac{1}{4}(M^2 - 5M + 8I)$

(b) $M^2 - 5M + 6I$

(c) $-\frac{1}{4}(M^2 - 5M + 4I)$

(d) $-\frac{1}{4}(M^2 + 5M + 8I)$

(e) $\frac{1}{4}(M^2 - M - 4I)$

10. Find the steady state probability vector of the following 3×3 stochastic matrix:

$$\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{2}{3} & 0 \end{bmatrix}$$

(a) $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{3}{8} \end{bmatrix}$ (c) $\begin{bmatrix} \frac{3}{8} \\ \frac{3}{8} \\ \frac{1}{4} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$ (e) $\begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

11. Which one of the following matrices is not diagonalisable, working over $\mathbb{C}$?

(a) $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

12. Which one of the following rules for $f : \mathbb{R}^2 \to \mathbb{R}^2$ defines a linear transformation?

(a) $f(x, y) = (x^2, y^2)$ (b) $f(x, y) = (y - x, x - y)$ (c) $f(x, y) = (y + 1, x + y)$

(d) $f(x, y) = (xy, y)$ (e) $f(x, y) = (2x, 3y + 4)$

13. Find the matrix corresponding to the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^3$ with the following rule:

$$f(x, y) = (6x - y, x + 2y, y - x)\ .$$

(a) $\begin{bmatrix} 6 & -1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 6 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 6 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} 6 & 1 & -1 \\ -1 & 2 & -1 \end{bmatrix}$

14. Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^4$ are linear transformations such that

$$M_f = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 3 \end{bmatrix} \quad \text{and} \quad M_g = \begin{bmatrix} 0 & 1 \\ 4 & 1 \\ 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Find the rule for the linear transformation $gf : \mathbb{R}^3 \to \mathbb{R}^4$.

(a) $(gf)(x, y, z) = (2x - 3y + 2z\ ,\ x - y + 8z\ ,\ -y - 4z\ ,\ 3x + 2y)$

(b) $(gf)(x, y, z) = (x + 9y + 2z\ ,\ -x - y\ ,\ 3x + 7y + z\ ,\ 3x - 2y)$

(c) $(gf)(x, y, z) = (x + y - 3z\ ,\ 9x + y - 7z\ ,\ 3x - y + 4z, -3x + y - 4z)$

(d) $(gf)(x, y, z) = (x - y + 3z\ ,\ 9x - y + 7z\ ,\ 2x + z\ ,\ -3x + y - 4z)$

(e) $(gf)(x, y, z) = (x - y + 3z\ ,\ 9x - y + 7z\ ,\ -3x + y - 4z\ ,\ 2x + z)$

15. Which of the following groups, under addition, is not cyclic?

(a) $\mathbb{Z}_3$ (b) $\mathbb{Z}_4$ (c) $\mathbb{Z}_2 \times \mathbb{Z}_3$ (d) $\mathbb{Z}_3 \times \mathbb{Z}_3$ (e) $\mathbb{Z}$

16. Consider the group G of symmetries of a regular pentagon, generated by a rotation α and a reflection β. Simplify the following expression in G:

$$\beta\alpha^3\beta^3\alpha^{-2}\beta^{-3}\alpha^8\beta$$

(a) α (b) $\alpha\beta$ (c) $\alpha^2\beta$ (d) α^2 (e) β

17. Consider the permutations

$$\alpha = (1\ 3\ 2)(4\ 6\ 5)(7\ 8) \quad \text{and} \quad \beta = (1\ 4\ 2)(8\ 6\ 3)(5\ 7)$$

of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ expressed in cycle notation. Which one of the following is a correct expression for the permutation

$$\gamma = \beta^{-1}\alpha\beta$$

where we compose from left to right?

(a) $\gamma = (4\ 6\ 2)(8\ 7\ 1)(5\ 3)$ (b) $\gamma = (5\ 6\ 8)(4\ 3\ 1)(7\ 2)$ (c) $\gamma = (2\ 6\ 4)(1\ 8\ 7)(3\ 5)$
(d) $\gamma = (7\ 3\ 2)(8\ 4\ 1)(6\ 5)$ (e) $\gamma = (2\ 3\ 7)(1\ 4\ 8)(6\ 5)$

18. Consider the permutations

$$\alpha = (1\ 3\ 2)(4\ 6\ 5)(7\ 8) \quad \text{and} \quad \beta = (1\ 4\ 2)(8\ 6\ 3)(5\ 7)$$

of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ expressed in cycle notation. Which one of the following is a correct expression for a permutation γ with the property

$$\beta = \gamma^{-1}\alpha\gamma$$

where we compose from left to right?

(a) $\gamma = (5\ 7\ 8\ 4\ 3)$ (b) $\gamma = (5\ 8\ 3\ 4)$ (c) $\gamma = (3\ 8\ 5\ 7\ 4)$
(d) $\gamma = (1\ 8\ 7\ 5\ 2\ 3\ 6\ 4)$ (e) $\gamma = (1\ 8\ 2\ 3\ 6)$

19. Which one of the following configurations is possible to reach from the 15-puzzle

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

by moving squares in and out of the space?

(a)

4	3	2	1
5	7	8	6
9	10	11	12
13	14	15	

(b)

15	2	3	13
5	6	10	8
9	7	11	12
4	14	1	

(c)

15	14	13	12
11	10	9	8
7	6	5	4
3	2	1	

(d)

14	12	3	15
5	2	10	8
13	7	11	6
9	4	1	

(e)

12	8	9	13
2	10	7	15
3	6	11	14
4	5	1	

20. Which one of the following configurations is impossible to reach from the 15-puzzle

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

by moving squares in and out of the space?

(a)

4	3	2	1
5	8	7	
9	10	11	12
13	14	15	6

(b)

1	2	3	4
5	6		8
9	10	11	12
13	14	15	7

(c)

1	8	9	
2	7	10	15
3	6	11	14
4	5	12	13

(d)

	3	6	15
5	2	10	8
13	7	11	12
9	4	1	14

(e)

	15	14	13
12	11	10	9
8	7	6	5
4	3	2	1