The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Exercises for Week 3 (beginning 11 March)** 2019

Important Ideas and Useful Facts:

(i) Systems of equations and augmented matrices: A *system* of m linear equations in n variables $x_1, x_2, \ldots, x_n$ over a field F has the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

where $a_{i,j}, b_i \in F$ are constants, for $1 \leq i \leq m$ and $1 \leq j \leq n$, with *augmented matrix*

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

The system is *homogeneous* if $b_1 = b_2 = \ldots = b_m = 0$.

(ii) Elementary row operations: There are three types of *elementary row operations* performed on (augmented) matrices:

(a) interchanging the ith and jth rows (denoted by $R_i \leftrightarrow R_j$)

(b) multiplying the ith row through by a nonzero constant λ (denoted by $R_i \to \lambda R_i$)

(c) adding a multiple of the jth row to the ith row (denoted by $R_i \to R_i + \lambda R_j$)

(iii) Row echelon form: A matrix is in *row echelon form* if

(a) rows of zeros appear at the bottom,

(b) first nonzero (*leading*) entries of consecutive rows appear further to the right,

(c) leading entries of rows are equal to 1;

and in *reduced row echelon form* if, in addition,

(d) entries above (and below) leading entries are zero.

(iv) Gaussian elimination and back substitution: The process of *Gaussian elimination* applies elementary row operations (*row reduction*) to transform the augmented matrix of a system into row echelon form, after which the associated system is solved using *back substitution*:

(a) the *leading variables* corresponding to leading entries are evaluated one equation at a time from the bottom towards the top,

(b) parameters are assigned to each nonleading variable (if any).

(v) Inconsistency: A system is inconsistent (that is, has no solution) if and only if at some stage in the process of row reduction a row of the following form is produced for some nonzero scalar k:

$$\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & k \end{array}$$

(vi) Uniqueness of reduced row echelon form: A given matrix can be row reduced to one and only one matrix in reduced row echelon form.

(vii) Injective, surjective and bijective functions: Consider a function $f : A \to B$ with domain A and codomain B. We say that f is *injective* if it is one-one, that is, different inputs yield different outputs,

$$(\forall x_1, x_2 \in A) \quad x_1 \neq x_2 \quad \text{implies} \quad f(x_1) \neq f(x_2) \ ,$$

or, equivalently (taking the contrapositive), the same outputs come from the same inputs,

$$(\forall x_1, x_2 \in A) \quad f(x_1) = f(x_2) \quad \text{implies} \quad x_1 = x_2 \ .$$

We say that f is *surjective* if it is onto, that is, every element of the codomain arises as a value of the function,

$$(\forall y \in B)(\exists x \in A) \quad y = f(x) \ .$$

We say that f is *bijective* if it is both injective and surjective. If f is bijective then the inverse function $f^{-1} : B \to A$ exists, whose rule is defined by undoing the rule for f, that is,

$$(\forall x \in A)(\forall y \in B) \quad y = f(x) \quad \text{if and only if} \quad x = f^{-1}(y) \ .$$

(viii) Permutations and the symmetric group: A *permutation* is a bijective function $f : X \to X$, where the domain and codomain coincide. The composition of two permutations is a permutation, and composition is often executed from left to right (preferred by most algebraists), rather than right to left (preferred by most analysts). Thus, if $\alpha : X \to X$ and $\beta : X \to X$ are permutations of X then the composite $\alpha\beta : X \to X$ is a permutation, where, using the algebraist's notation,

$$(\forall x \in X) \quad x(\alpha\beta) \ = \ ((x\alpha)\beta) \ .$$

We call

$$S_X \ = \ \{\text{permutations of } X\}$$

the *symmetric group on* X, which becomes a group with respect to composition of permutations. The identity element of S_X is the *identity permutation* that fixes all elements of X, and inverses in the group are the usual inverses as bijections. If $X = \{1, \ldots, n\}$, where $n \geq 1$, then it is usual to write S_n for S_X.

(ix) Cycles and cycle notation: Let $\phi : X \to X$ be a permutation. Elements of X are called *letters*. Call ϕ a *cycle* if every letter that is moved by ϕ can be reached from every other letter that is moved by ϕ, by repeating ϕ often enough. Every permutation can be expressed as a composite (product) of *disjoint* cycles, that is, cycles where the sets of letters that are moved by the cycles are disjoint. To express ϕ using cycle notation, perform the following steps (assuming X is nonempty):

(a) Write a left bracket followed by any letter of X.

(b) Write down letters in order as produced by applying ϕ, until the first repetition, closing off with a right bracket.

(c) Choose a new letter and repeat, until all letters of X have been exhausted.

(d) Ignore singleton cycles (and denote the identity permutation by 1).

Questions labelled with an asterisk are suitable for students aiming for a distinction or higher.

Tutorial Exercises:

1. Look at the corner of the room. The walls are two planes meeting in a line. Follow the line to the ceiling. Where it meets the ceiling is the point of intersection of the three planes. Find the point $(x, y, z) \in \mathbb{R}^3$ of intersection of the following three planes:

$$\begin{array}{rcrcrcr} 2x & + & 3y & + & 4z & = & -4 \\ 5x & + & 5y & + & 6z & = & -3 \\ 3x & + & y & + & 2z & = & -1 \end{array}$$

2. Solve the following systems of linear equations over $\mathbb{R}$ and over $\mathbb{Z}_7$:

(a)
$$\begin{array}{rcrcrcr} 2x & - & 2y & + & z & = & 0 \\ x & + & y & + & 2z & = & 0 \\ -3x & + & y & + & 4z & = & 0 \end{array}$$

(b)
$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & - & 2x_2 & + & 2x_3 & = & -2 \end{array}$$

(c)
$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 2 \\ x_1 & - & x_2 & + & x_3 & & & - & x_5 & = & 0 \\ x_1 & + & 5x_2 & + & x_3 & + & 3x_4 & + & 3x_5 & = & 3 \end{array}$$

3.* Working over $\mathbb{R}$ and $\mathbb{Z}_5$, find the respective values of λ such that the following system is inconsistent, has more than one solution, or has a unique solution:

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & -3 \\ -2x & - & \lambda y & + & z & = & 2 \\ x & + & 2y & + & \lambda z & = & 1 \end{array}$$

4. Express the following permutations of $\{1, 2, 3, 4, 5, 6\}$ using cycle notation:

$$\alpha \ : \ 1 \mapsto 3,\ 2 \mapsto 5,\ 3 \mapsto 6,\ 4 \mapsto 2,\ 5 \mapsto 1,\ 6 \mapsto 4$$

$$\beta \ : \ 1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 3,\ 4 \mapsto 6,\ 5 \mapsto 5,\ 6 \mapsto 4$$

$$\gamma \ : \ 1 \mapsto 4,\ 2 \mapsto 3,\ 3 \mapsto 6,\ 4 \mapsto 5,\ 5 \mapsto 1,\ 6 \mapsto 2$$

5. Find the composites $\alpha\beta$, $\alpha\gamma$ and $\beta\gamma$ where α, β and γ are the permutations defined in the previous question. Express the inverses of α, β, γ as positive powers of α, β, γ respectively.

6.* Recall that $\mathbb{Z}_2 = \{0, 1\}$ with usual addition and multiplication except that $1+1 = 0$.

(a) List all 2×2 matrices with entries from $\mathbb{Z}_2$ that have nonzero determinant (so are invertible).

(b) A column vector is called *nonzero* if at least one of its entries is nonzero. Let A be an invertible $n \times n$ matrix. Prove that $\phi : \mathbf{v} \mapsto A\mathbf{v}$ is a permutation of $X = \{\, \mathbf{v} \mid \mathbf{v}$ is a nonzero column vector with n entries $\}$.

(c) Draw an equilateral triangle and label the vertices (in any order) by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

By (b) each of the matrices in (a) permutes $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ by premultiplication. Describe the geometric effect on the triangle achieved in each case.

Further Exercises:

7. For each of the following augmented matrices, working over $\mathbb{R}$ and $\mathbb{Z}_5$, decide whether the system of equations to which it corresponds has no solution, a unique solution, finitely many solutions (but more than one), or infinitely many solutions.

(a) $\left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 1 & 2 & -2 & 0 \\ -2 & -4 & 4 & 0 \end{array}\right]$ (b) $\left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ -1 & -2 & 2 & 0 \\ 1 & 3 & 3 & 3 \end{array}\right]$

(c) $\left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 1 & 2 & -2 & 3 \\ -1 & -2 & 2 & 2 \end{array}\right]$ (d) $\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 3 & 1 & -1 \\ -2 & -2 & 4 & 3 & 1 & 0 \\ 0 & 0 & 0 & -3 & -1 & 4 \end{array}\right]$

8. Find constants A, B, C, D that satisfy the following:

$$\frac{x^3}{(x-1)^4} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4}$$

What happens if the coefficients of the polynomials come from the fields $\mathbb{Z}_3$ or $\mathbb{Z}_2$?

9. A real cubic polynomial $p(x)$ with derivative $p'(x)$ satisfies the following conditions:

$$\text{(a)} \quad p(1) = p'(1) = 4\,, \qquad \text{(b)} \quad p(2) = 14\,, \qquad \text{(c)} \quad p'(2) = 17\,.$$

Find $p(x)$. What happens, in each case, if condition (c) is removed, condition (b) is removed, or condition (a) is removed?

10.* Use row reduction on augmented matrices to show that the lines $ax + by = k$ and $cx + dy = \ell$ in the real plane intersect in a single point if and only if $ad - bc \neq 0$.

11.* Let $\alpha : 1 \mapsto 2 \mapsto 3 \mapsto 1$ and $\beta : 1 \mapsto 1,\ 2 \mapsto 3 \mapsto 2$. Verify the following equations (where 1 denotes the identity permutation):

$$\alpha^3 = \beta^2 = 1\,, \quad \beta\alpha = \alpha^2\beta\,, \quad \beta\alpha\beta = \alpha^{-1}\,.$$

Put $G = \langle \alpha, \beta \rangle$, the group generated by α and β. Deduce that

$$G = \{\, \alpha^i \beta^j \mid 0 \le i \le 2\,,\ 0 \le j \le 1 \,\}\,.$$

If the letters $1, 2, 3$ are vertices of an equilateral triangle, interpret elements of G in terms of their geometric effects on the triangle. How many different interpretations of the symbol 1 appear in this exercise? Are you bothered by this?

12.* Let $\phi, \psi : \mathbb{R}\backslash\{0,1\} \to \mathbb{R}\backslash\{0,1\}$ where, for each $x \in \mathbb{R}\backslash\{0,1\}$,

$$x\phi = \frac{1}{1-x}\,, \quad x\psi = \frac{1}{x}\,.$$

Verify that ϕ and ψ are sensibly defined permutations and

$$\phi^3 = \psi^2 = 1\,, \quad \psi\phi = \phi^2\psi\,.$$

Put $H = \langle \phi, \psi \rangle$. How does H relate to G of the previous question?