The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Week 3 Longer Solutions** 2019

1. Avoiding fractions:

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & -4 \\ 5 & 5 & 6 & -3 \\ 3 & 1 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & 3 \\ 1 & -1 & -2 & 5 \\ 3 & 1 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 7 & 8 & -10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 8 & -24 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

giving the unique intersection point $(1, 2, -3)$.

2. (a) Working over $\mathbb{R}$:

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ -3 & 1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -4 & -3 & 0 \\ 0 & 4 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -4 & -3 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right]$$

giving immediately the unique solution $(0, 0, 0)$. Continuing working over $\mathbb{Z}_7$ the last matrix simplifies to become

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

giving the solution set $\{(4t, t, t) \mid t \in \mathbb{Z}_7\}$.

(b) Working over $\mathbb{R}$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 1 \end{array}\right]$$

giving the solution $\{(-\frac{4t}{3}, 1 + \frac{t}{3}, t) \mid t \in \mathbb{R}\}$. Working over $\mathbb{Z}_7$ this simplifies to yield the solution $\{(t, 1 + 5t, t) \mid t \in \mathbb{Z}_7\}$.

(c) Working over $\mathbb{R}$:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & 5 & 1 & 3 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 1 & 2 & 2 \\ 0 & 4 & 0 & 2 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 & -3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & \frac{3}{2} \end{array}\right]$$

giving the solution $\{(1-s-\frac{t}{2}, -\frac{1}{2}-\frac{t}{2}, s, t, \frac{3}{2}) \mid s, t \in \mathbb{R}\}$. Working over $\mathbb{Z}_7$ this simplifies to yield the solution $\{(1 - s + 3t, 3 + 3t, s, t, 5) \mid s, t \in \mathbb{Z}_7\}$.

3. Working over $\mathbb{R}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ -2 & -\lambda & 1 & 2 \\ 1 & 2 & \lambda & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & -\lambda & -5 & -4 \\ 0 & 2 & \lambda+3 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & \frac{\lambda+3}{2} & 2 \\ 0 & \lambda & 5 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & \frac{\lambda+3}{2} & 1 \\ 0 & 0 & 5-\frac{\lambda(\lambda+3)}{2} & 4-2\lambda \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & \frac{\lambda+3}{2} & 1 \\ 0 & 0 & 10-\lambda(\lambda+3) & 4(2-\lambda) \end{array}\right]$$

Observe that $10-\lambda(\lambda+3) = 10-3\lambda+\lambda^2 = (\lambda+5)(2-\lambda)$, so we get a unique solution when this is nonzero, that is, when $\lambda \neq 2, -5$.

We get an infinite solution when $(\lambda+5)(2-\lambda) = 4(2-\lambda) = 0$, that is, when $\lambda = 2$.

We get no solution when $(\lambda+5)(2-\lambda)0$ but $4(2-\lambda) \neq 0$, that is, when $\lambda = -5$.

Working over $\mathbb{Z}_5$, the row reductions are all valid and the echelon forms are the same, except that $5 = 0$. Thus, we get a unique solution when $\lambda \neq 0, 2$, that is, $\lambda \in \{1, 3, 4\}$, a one-parameter family of five solutions when $\lambda = 2$, and no solution when $\lambda = 0$.

4. We have $\alpha = (136425)$, $\beta = (12)(46)$ and $\gamma = (145)(236)$.

5. We have $\alpha\beta = (134)(25)$, $\alpha\gamma = (165432)$, $\beta\gamma = (1365)(24)$, $\alpha^{-1} = \alpha^5$, $\beta^{-1} = \beta$ and $\gamma^{-1} = \gamma^2$.

6. (a) The determinant is nonzero when both diagonal [off-diagonal] entries are 1 and at least one of the off-diagonal [diagonal] entries is 0, leading to the following possibilities:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(b) If $A\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = I\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}\mathbf{0} = \mathbf{0}$. This demonstrates that

$$\mathbf{v} \neq \mathbf{0} \quad \Longrightarrow \quad A\mathbf{v} \neq \mathbf{0}\,. \qquad (*)$$

Hence indeed $\phi : X \to X$. If $\mathbf{v} \in X$ then, since A^{-1} is also invertible, the same argument shows $A^{-1}\mathbf{v} \in X$; also $\mathbf{v} = 1\mathbf{v} = AA^{-1}\mathbf{v} = \left(A^{-1}\mathbf{v}\right)\phi$. This verifies that ϕ is onto. If $\mathbf{v}_1, \mathbf{v}_2 \in X$ and $\mathbf{v}_1\phi = \mathbf{v}_2\phi$ then $A\mathbf{v}_1 = A\mathbf{v}_2$, so that $A(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$, from which it follows, by $(*)$, that $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$, which shows $\mathbf{v}_1 = \mathbf{v}_2$. Hence ϕ is also one-one. This proves ϕ is a permutation of X.

(c) Certainly the matrix I fixes the triangle (a rotation of 0 radians). Observe that

$$A\mathbf{v}_1 = \mathbf{v}_2\,, \quad A\mathbf{v}_2 = \mathbf{v}_3\,, \quad A\mathbf{v}_3 = \mathbf{v}_1\,,$$

$$B\mathbf{v}_1 = \mathbf{v}_3\,, \quad B\mathbf{v}_3 = \mathbf{v}_2\,, \quad B\mathbf{v}_2 = \mathbf{v}_1\,,$$

so A and B are rotations of the triangle $2\pi/3$ and $4\pi/3$ radians respectively. Further

$$C\mathbf{v}_1 = \mathbf{v}_2\,, \quad C\mathbf{v}_2 = \mathbf{v}_1\,, \quad C\mathbf{v}_3 = \mathbf{v}_3\,,$$

$$D\mathbf{v}_1 = \mathbf{v}_3\,, \quad D\mathbf{v}_2 = \mathbf{v}_2\,, \quad D\mathbf{v}_3 = \mathbf{v}_1\,,$$

$$E\mathbf{v}_1 = \mathbf{v}_1\,, \quad E\mathbf{v}_2 = \mathbf{v}_3\,, \quad E\mathbf{v}_3 = \mathbf{v}_2\,,$$

so C, D and E correspond to reflections of the triangle in respective axes of symmetry. (We are in fact reproducing the symmetry group of the triangle, up to isomorphism.)

7. (a) Working over $\mathbb{R}$:

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 1 & 2 & -2 & 0 \\ -2 & -4 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 0 & 0 & -5 & -3 \\ 0 & 0 & 10 & 6 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

yielding an infinite solution (using one parameter). Working over $\mathbb{Z}_5$:

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 1 & 2 & -2 & 0 \\ -2 & -4 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & -3 \\ -2 & -4 & 4 & 0 \end{array}\right]$$

yielding an inconsistent system, with no solutions.

(b) Working over $\mathbb{R}$:

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ -1 & -2 & 2 & 0 \\ 1 & 3 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 0 & 0 & 5 & 3 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 3 \end{array}\right]$$

yielding a unique solution. Working over $\mathbb{Z}_5$:

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ -1 & -2 & 2 & 0 \\ 1 & 3 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 3 \\ 1 & 3 & 3 & 3 \end{array}\right]$$

yielding an inconsistent system with no solutions.

(c) Working over $\mathbb{R}$:

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 1 & 2 & -2 & 3 \\ -1 & -2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 5 & 5 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

yielding an inconsistent system with no solutions. Working over $\mathbb{Z}_5$:

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 1 & 2 & -2 & 3 \\ -1 & -2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

yielding a finite solution set with $5^2 = 25$ solutions (using two parameters).

(d) Working over $\mathbb{R}$:

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 3 & 1 & -1 \\ -2 & -2 & 4 & 3 & 1 & 0 \\ 0 & 0 & 0 & -3 & -1 & 4 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 3 & 1 & -1 \\ 0 & 0 & 0 & 9 & 3 & -2 \\ 0 & 0 & 0 & 3 & 1 & -4 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 3 & 1 & -1 \\ 0 & 0 & 0 & 3 & 1 & -4 \\ 0 & 0 & 0 & 0 & 10 & 0 \end{array}\right]$$

yielding an inconsistent system with no solutions. Working over $\mathbb{Z}_5$, the last row becomes all zero, and we get a finite solution set with $5^3 = 125$ solutions (using three parameters).

8. Multiplying through by $(x-1)^4$ we get the following polynomial equation:

$$x^3 \;=\; A(x-1)^3 + B(x-1)^2 + C(x-1) + D$$

In particular, this must hold for all x from the underlying field. Working over $\mathbb{R}$ we can substitute $x = 1$ to get $D = 1$ immediately. Substituting $x = 0$, $x = 2$ and $x = -1$ gives

$$\begin{aligned} 0 &= -A+B-C+D \;=\; -A+B-C+1 \\ 8 &= A+B+C+D \;=\; A+B+C+1 \\ -1 &= -8A+4B-2C+D \;=\; -8A+4B-2C+1 \end{aligned}$$

which rearrange to give the following system:

$$\begin{array}{rcrcrcl} A & - & B & + & C & = & 1 \\ A & + & B & + & C & = & 7 \\ 8A & - & 4B & + & 2C & = & 2 \end{array}$$

which we can solve as follows:

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 7 \\ 8 & -4 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 6 \\ 0 & 4 & -6 & -6 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -6 & -18 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

yielding $A = 1$, $B = 3$, $C = 3$, $D = 1$. In particular we have the following equation involving polynomials with integer coefficients:

$$x^3 \;=\; (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1\,.$$

If we interpret this equation modulo 3 we get

$$x^3 \;=\; (x-1)^3 + 1\,.$$

Thus, over $\mathbb{Z}_3$, we get $A = 1$, $B = 0$, $C = 0$, $D = 1$.

If we interpret the equation over the integers modulo 2 we get

$$x^3 \;=\; (x+1)^3 + (x+1)^2 + (x+1) + 1\,.$$

Thus, over $\mathbb{Z}_2$, we get $A = B = C = D = 1$.

9. We have $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ for some constants a_0, a_1, a_2, a_3, so that

$$p'(x) \;=\; a_1 + 2a_2x + 3a_3x^2\,.$$

Conditions (a), (b), (c) provide the following system of equations:

$$\begin{array}{rcrcrcrcl} a_0 & + & a_1 & + & a_2 & + & a_4 & = & 4 \\ & & a_1 & + & 2a_2 & + & 3a_3 & = & 4 \\ a_0 & + & 2a_1 & + & 4a_2 & + & 8a_4 & = & 14 \\ & & a_1 & + & 4a_2 & + & 12a_3 & = & 17 \end{array}$$

which we can solve as follows:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 & 14 \\ 0 & 1 & 4 & 12 & 17 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 7 & 10 \\ 0 & 0 & 2 & 9 & 13 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 6 \\ 0 & 1 & 0 & -5 & -8 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

yielding $a_3 = 1$, $a_2 = 2$, $a_1 = -3$, $a_0 = 4$. Thus

$$p(x) \;=\; 4 - 3x + 2x^2 + x^3 \,.$$

If condition (c) is removed then we get the system

$$\begin{array}{rcrcrcrcr} a_0 & + & a_1 & + & a_2 & + & a_4 & = & 4 \\ & & a_1 & + & 2a_2 & + & 3a_3 & = & 4 \\ a_0 & + & 2a_1 & + & 4a_2 & + & 8a_4 & = & 14 \end{array}$$

which we solve as follows:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 & 14 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 7 & 10 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 6 \\ 0 & 1 & 0 & -5 & -8 \\ 0 & 0 & 1 & 4 & 6 \end{array}\right]$$

which yields the parametric solution $a_3 = t$, $a_2 = 6 - 4t$, $a_1 = -8 + 5t$, $a_0 = 6 - 2t$ for $t \in \mathbb{R}$. Thus the polynomials have the form

$$p(x) \;=\; 6 - 2t + (5t - 8)x + (6 - 4t)x^2 + tx^3 \,.$$

If condition (b) is removed then we get the system

$$\begin{array}{rcrcrcrcr} a_0 & + & a_1 & + & a_2 & + & a_4 & = & 4 \\ & & a_1 & + & 2a_2 & + & 3a_3 & = & 4 \\ & & a_1 & + & 4a_2 & + & 12a_3 & = & 17 \end{array}$$

which we can solve as follows:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 12 & 17 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 9 & 13 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & \frac{9}{2} & \frac{13}{2} \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{5}{2} & \frac{13}{2} \\ 0 & 1 & 0 & -6 & -9 \\ 0 & 0 & 1 & \frac{9}{2} & \frac{13}{2} \end{array}\right]$$

which yields the parametric solution $a_3 = t$, $a_2 = \frac{13-9t}{2}$, $a_1 = 6t - 9$, $a_0 = \frac{13-5t}{2}$ for $t \in \mathbb{R}$. Thus the polynomials have the form

$$p(x) = \frac{13-5t}{2} + (6t-9)x + \frac{13-9t}{2}x^2 + tx^3 .$$

If condition (a) is removed then we get the system

$$\begin{array}{rcrcrcrcl} a_0 & + & 2a_1 & + & 4a_2 & + & 8a_4 & = & 14 \\ & & a_1 & + & 4a_2 & + & 12a_3 & = & 17 \end{array}$$

which we can solve as follows:

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 14 \\ 0 & 1 & 4 & 12 & 17 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -4 & -16 & -20 \\ 0 & 1 & 4 & 12 & 17 \end{array}\right]$$

yielding the doubly parametric solution $a_3 = t$, $a_2 = s$, $a_1 = 17 - 4s - 12t$, $a_0 = -20 + 4s + 16t$ for $t \in \mathbb{R}$. Thus the polynomials have the form

$$p(x) = -20 + 4s + 16t + (17 - 4s - 12t)x + sx^2 + tx^3 .$$

10. Suppose first that $a \neq 0$. Then the system has the augmented matrix

$$\left[\begin{array}{cc|c} a & b & k \\ c & d & \ell \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & b/a & k/a \\ c & d & \ell \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & b/a & k/a \\ 0 & d - \frac{bc}{a} & \ell - \frac{ck}{a} \end{array}\right]$$

which yields a unique solution if and only if $d - \frac{bc}{a} = \frac{ad-bc}{a} \neq 0$, which occurs if and only if $ad - bc \neq 0$. Suppose now that $a = 0$. The system now has the augmented matrix

$$\left[\begin{array}{cc|c} 0 & b & k \\ c & d & \ell \end{array}\right] \sim \left[\begin{array}{cc|c} c & d & \ell \\ 0 & b & k \end{array}\right]$$

which yields a unique solution if and only if $c \neq 0$ and $b \neq 0$, which occurs if and only if $ad - bc = -bc \neq 0$. Thus in all cases there is a unique solution, that is a unique intersection point of the two lines, if and only if $ad - bc \neq 0$.

11. We have $\alpha = (123)$ and $\beta = (23)$ so clearly $\alpha^3 = \beta^2 = 1$. Note that $\alpha^{-1} = \alpha^2$ and $\beta^{-1} = \beta$. Further

$$\beta\alpha = (23)(123) = (21) = (12) = (123)(123)(23) = \alpha^2\beta ,$$

from which it follows that $\beta\alpha\beta = \alpha^2\beta^2 = \alpha^2 1 = \alpha^2 = \alpha^{-1}$. Now $G = \langle \alpha, \beta \rangle$ comprises all permutations that can be obtained by combining α and β any number of times in any order. But the equation $\beta\alpha = \alpha^2\beta$ can be used to sort all α's to the left and β's to the right. Then the equations $\alpha^3 = \beta^2 = 1$ can be used to replace any power of α by 1, α or α^2 and any power of β by 1 or β. Thus a general expression involving α's and β's becomes $\alpha^i\beta^j$ for some $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$. This proves

$$G \subseteq \{ \alpha^i\beta^j \mid 0 \leq i \leq 2 , \; 0 \leq j \leq 1 \} .$$

But the set on the right clearly is a subset of G, so we get equality. The geometrical effects are that 1, α and α^2 are rotations of the equilateral triangle 0, $2\pi/3$ and $4\pi/3$

radians respectively and β, $\alpha\beta$ and $\alpha^2\beta$ are reflections in the three respective axes of symmetry of the triangle.

The symbol 1 appears in five guises: (i) as a letter; (ii) as a permutation; (iii) as an integer; (iv) as the label on an imaginary diagram; (v) as a symbol whose meanings are being discussed by this question. No, *one* should not bothered by this at all: 1 to many relationships are ubiquitous in natural language, without which communication would become impossibly congested.

12. Observe that $\frac{1}{x}$ is never zero and does not equal 1 if $x \neq 1$, which shows the range of ψ is indeed contained in $\mathbb{R}\backslash\{0,1\}$. Suppose $y \in \mathbb{R}\backslash\{0,1\}$. Then $\frac{1}{y} \in \mathbb{R}\backslash\{0,1\}$ and $\left(\frac{1}{y}\right)\psi = y$, which shows ψ is onto. Further, if $x_1, x_2 \in \mathbb{R}\backslash\{0,1\}$ and $x_1\psi = x_2\psi$ then $\frac{1}{x_1} = \frac{1}{x_2}$, so $x_1 = x_2$, which verifies that ψ is one-one. Hence ψ is a permutation of $\mathbb{R}\backslash\{0,1\}$. Similarly ϕ is a permutation of $\mathbb{R}\backslash\{0,1\}$ (or alternatively observe that ϕ is the composite with ψ of the permutation: $x \mapsto 1-x$.) Now

$$x\phi^2 \;=\; \left(\frac{1}{1-x}\right)\phi \;=\; \frac{1}{1-\frac{1}{1-x}} \;=\; \frac{1-x}{1-x-1} \;=\; \frac{x-1}{x} \;=\; 1-\frac{1}{x}\,,$$

so that

$$x\phi^3 \;=\; (x\phi^2)\phi \;=\; \left(1-\frac{1}{x}\right)\phi \;=\; \frac{1}{\frac{1}{x}} \;=\; x\,.$$

This proves $\phi^3 = 1$. It is immediate also that $\psi^2 = 1$. Observe that

$$x\psi\phi \;=\; \left(\frac{1}{x}\right)\phi \;=\; \frac{1}{1-\frac{1}{x}} \;=\; \frac{x}{x-1}$$

and

$$x\phi^2\psi \;=\; \left(\frac{x-1}{x}\right)\psi \;=\; \frac{x}{x-1}\,,$$

which proves $\psi\phi = \phi^2\psi$. The equations relating ϕ and ψ are therefore the same as in the previous question, so

$$H \;=\; \langle\phi,\psi\rangle \;=\{\,\phi^i\psi^j \mid 0\le i\le 2\,,\; 0\le j\le 1\,\}\,.$$

Observe that there is no collapse:

$$1 : x \mapsto x\,, \qquad \alpha : x \mapsto \frac{1}{1-x}\,, \qquad \alpha^2 : x \mapsto \frac{x-1}{x}$$

$$\beta : x \mapsto \frac{1}{x}\,, \qquad \alpha\beta : x \mapsto 1-x\,, \qquad \alpha^2\beta : x \mapsto \frac{x}{x-1}$$

are 6 distinct permutations, so the multiplication tables of the groups H and G look the same if one renames α by ϕ and β by ψ. (In the language of abstract groups – see later in the semester – the groups are *isomorphic*).