The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Exercises for Week 4 (beginning 18 March)** 2019

Important Ideas and Useful Facts:

(i) Elementary matrices: An $n \times n$ matrix is called *elementary* if it is the result of applying a single elementary row operation to the identity matrix I_n.

(ii) Effect of multiplication by an elementary matrix: If E is the elementary matrix obtained by applying the elementary row operation ρ to I_n, and A is any matrix with n rows, then the matrix product EA is the matrix obtained by applying ρ to A.

(iii) Invertibility criterion and algorithm: A square matrix A is invertible if and only if A is a product of elementary matrices, which occurs if and only if the augmented matrix $[\,A \mid I\,]$ can be row reduced to $[\,I \mid B\,]$, in which case $A^{-1} = B$.

(iv) Half of the definition of invertibility suffices for square matrices: If A is a square matrix and $AB = I$ or $BA = I$ then $AB = BA = I$, in which case the inverse A^{-1} exists and equals B.

(v) Determinants of matrices of dimensions 1, 2 and 3: The *determinant* of a 1×1 matrix $[a]$ is simply the entry a. The *determinant* of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. The *determinant* of a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ is

$$\det A = |A| = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

called the *expansion along the first row*, where the smaller determinant arises by ignoring the row and column of the entry being used as a coefficient.

(vi) Determinants in general: Following the pattern for 3×3 matrices, we may expand along any row or down any column of a given square matrix A of any size, producing the same number, called the *determinant* of A, denoted by $\det A$ or $|A|$, provided one uses adjustment factors given by the chequerboard patterns

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} , \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

and so on to higher dimensions.

(vii) Multiplicative property of determinants: If A and B are square matrices of the same size then $\det(AB) = (\det A)(\det B)$.

(viii) Invertibility criterion using determinants: A square matrix is invertible if and only if its determinant is nonzero.

(ix) Effects of elementary row and columns operations on determinants: Let A be a square matrix. If B is obtained from A by swapping two rows or swapping two columns then

$$\det B \;=\; -\det A\;.$$

If B is obtained from A by multiplying a row or column by a scalar λ then

$$\det B \;=\; \lambda \det A\;.$$

If B is obtained from A by adding a multiple of one row [column] to another row [column] then

$$\det B \;=\; \det A\;.$$

(x) Determinant of the transpose: If A is a square matrix then $\det(A^T) = \det A$.

(xi) Transpositions: A permutation that interchanges two letters and fixes all other letters is called a *transposition*.

(xii) Even and odd permutations: A permutation of a finite set is called *even* if it is a product of an even number of transpositions (and by default the identity permutation is even), and called *odd* if it is a product of an odd number of transpositions.

(xiii) No permutation can be both even and odd: If n is a positive integer then the symmetric group S_n is the disjoint union of A_n, the subset of even permutations (and called the *alternating group*), and $S_n \backslash A_n$, the complement of A_n, which comprises exactly the subset of all odd permutations.

(xiv) Conjugates of permutations: If α and β are any permutations of a given set then the *conjugate of* α *by* β is the permutation $\beta^{-1}\alpha\beta$, denoted by α^β, using exponential notation. We denote by $\alpha^{-\beta}$ the inverse of α^β, so that $\alpha^{-\beta} = \beta^{-1}\alpha^{-1}\beta$, the conjugate of α^{-1} by β (not to be confused with $\beta\alpha\beta^{-1}$, which is $\alpha^{\beta^{-1}}$, the congugate of α by β^{-1}).

(xv) Effect of conjugation on the cycle decomposition of a permutation: If α and β are permutations and $(a_1\ a_2\ \ldots\ a_k)$ is a cycle in the cycle decomposition of α then $(a_1\beta\ a_2\beta\ \ldots\ a_k\beta)$ is a cycle in the cycle decomposition of the conjugate α^β.

Questions labelled with an asterisk are suitable for students aiming for a distinction or higher.

Tutorial Exercises:

1. Find the inverse of $\begin{bmatrix} 5 & -3 \\ 7 & -4 \end{bmatrix}$ and use it to solve for x, y, z, w over $\mathbb{R}$ where

$$\begin{bmatrix} 5 & -3 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \;=\; \begin{bmatrix} 11 & 4 \\ 15 & 5 \end{bmatrix}$$

Explain why the solution also holds interpreted over $\mathbb{Z}_p$ for any prime number p.

2. Find the inverse of each of the following matrices if it exists, working over $\mathbb{R}$, over $\mathbb{Z}_2$ and over $\mathbb{Z}_3$:

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. Find $\det M$ where $M = \begin{bmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{bmatrix}$ by (a) expanding along the first row. Confirm your answer by (b) expanding along the second row, and also by (c) expanding down the third column. Can you find the answer faster by using elementary row or column operations? Explain briefly why M is not invertible if interpreted as a matrix over $\mathbb{Z}_{13}$, but M is invertible over $\mathbb{Z}_{11}$.

4. Write down the inverses of the following permutations as products of disjoint cycles:

$$\alpha = (1\,2\,3\,4\,5\,6)\,,\quad \beta = (1\,2)(3\,4)(5\,6\,7\,8)\,,\quad \gamma = (1\,3\,5)(2\,4\,6)$$

Express each of α, β, γ as a product of transpositions, and decide whether each is an even or an odd permutation.

5. Let α and β be permutations of a finite set. Recall that if $(x_1\ x_2\ \dots\ x_k)$ is a cycle in the decomposition of α then $(x_1\beta\ x_2\beta\ \dots\ x_k\beta)$ is a cycle in the decomposition of $\beta^{-1}\alpha\beta$. Use this fact to write down $\beta^{-1}\alpha\beta$ immediately when

(a) $\alpha = (1\,2\,3\,4\,5\,6)\,,\quad \beta = (1\,6)(2\,5)(3\,4)\,.$

(b) $\alpha = (1\,6)(2\,5)(3\,4)\,,\quad \beta = (1\,2\,3\,4\,5\,6)\,.$

(c) $\alpha = (1\,9)(4\,2\,5\,6)(8\,3\,7)\,,\quad \beta = (2\,3\,8\,6\,5\,4)(9\,1)\,.$

(d) $\alpha = (2\,3\,8\,6\,5\,4)(9\,1)\,,\quad \beta = (1\,9)(4\,2\,5\,6)(8\,3\,7)\,.$

6. Express each of the following as products of disjoint cycles and decide which are even or odd.

$$\alpha = (1\,2)(1\,4)\,,\quad \beta = (1\,2\,3)(1\,4\,5)\,,\quad \gamma = (1\,2\,3\,4)(4\,5\,6)(2\,4)$$

$$\delta = (1\,2\,3\,4\,5)(1\,3\,5)(2\,5\,6\,7)\,,\quad \varepsilon = (1\,2)(1\,3)(1\,4\,5)(4\,2)(3\,5\,7)$$

7.* It is a theorem that a configuration of the 8-puzzle with the bottom right square blank is possible if and only if the permutation associated with the other squares is even. Use this to decide which of the following configurations are impossible to reach:

(a)

2	1	3
8	4	5
7	6	

(b)

1	2	3
7	6	8
4	5	

(c)

7	1	2
3	5	6
4	8	

(d)

	1	4
3	2	7
6	5	8

(e)

3	1	4
2	8	
6	5	7

(f)

8	7	6
1		5
2	3	4

Further Exercises:

8. Find M^{-1} over $\mathbb{Z}_{11}$ where M is the matrix given in Exercise 3, which we know in advance is invertible.

9. Use elementary row and column operations to find the following determinants:

(a) $\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 3 \\ 4 & 5 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & 3 & 6 & 2 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{vmatrix}$

Explain briefly why the matrix for part (b) can never be invertible, but the matrix for (c) is invertible over $\mathbb{R}$ and also over $\mathbb{Z}_p$ for all odd primes p.

10. Explain briefly why a square matrix with two identical rows or two identical columns has zero determinant.

11. A square matrix A is called *idempotent* if $A^2 = A$ and *nilpotent* if $A^k = 0$, the zero matrix, for some positive integer k. Use the multiplicative property of the determinant to find all possible determinants of matrices that are (a) idempotent, or (b) nilpotent.

12. Let α be a permutation of $\{1, 2, \ldots, n\}$ where $n \geq 2$. Explain why $\alpha^{-1} = \alpha^k$ for some positive integer k. Explain briefly why the set of all odd permutations of $\{1, 2, \ldots, n\}$ does not form a permutation group.

13. Find all even permutations of $\{1, 2, 3, 4\}$ writing your answers using cycle notation.

14.* The collection of even permutations of $\{1, 2, 3, 4\}$ that you found in the previous exercise forms the alternating group A_4. A *subgroup* of A_4 is a nonempty subset closed under composition and inversion. Find a subgroup of A_4 for each of the sizes 1, 2, 3, 4 and 12. (It is a general fact known as Lagrange's Theorem, which you will meet later in the semester, that the size of a subgroup divides the size of the larger group.) Show, however, that no subgroup of A_4 exists of size 6.

15.* Square matrices A and B are said to be *conjugate* or *similar* if there exists an invertible matrix P such that $B = P^{-1}AP$.

(a) Verify that conjugacy is an equivalence relation on square matrices, that is, conjugacy is reflexive, symmetric and transitive.

(b) Use the multiplicative property of the determinant to verify that if A and B are conjugate then $\det A = \det B$.

(c) Verify that if A and B are conjugate matrices over a field F then $A - \lambda I$ and $B - \lambda I$ are also conjugate for all $\lambda \in F$. Deduce that $\det(A - \lambda I) = 0$ if and only if $\det(B - \lambda I) = 0$ for all $\lambda \in F$. (This proves that similar matrices have the same *eigenvalues*.)

16.* Use matrix arithmetic and determinants to give a quick proof that the lines $ax + by = k$ and $cx + dy = \ell$ in the plane $\mathbb{R}^2$ intersect in a single point if and only if $ad - bc \neq 0$.