The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Week 4 Longer Solutions** 2019

1. Put $M = \begin{bmatrix} 5 & -3 \\ 7 & -4 \end{bmatrix}$ and the equation becomes $M \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ 15 & 5 \end{bmatrix}$. Then

$$M^{-1} = \frac{1}{5(-4)-(-3)(7)} \begin{bmatrix} -4 & 3 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -7 & 5 \end{bmatrix},$$

so

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = M^{-1} \begin{bmatrix} 11 & 4 \\ 15 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 11 & 4 \\ 15 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix},$$

yielding $x = 1$, $y = -1$, $z = -2$, $w = -3$.

This solution will hold when all entries are interpreted over $\mathbb{Z}_p$, for any prime p, because the determinant of M as an integer is $5(-4) - (-3)(7) = 1$, which evaluates to $1 \neq 0$ in $\mathbb{Z}_p$, so that the formula for M^{-1} is always valid.

2. (a) Working over $\mathbb{R}$:

$$A^{-1} = \frac{1}{1(1)-1(0)} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

This becomes $A^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ over $\mathbb{Z}_2$, and $A^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ over $\mathbb{Z}_3$.

(b) Row reducing over $\mathbb{R}$:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

giving

$$B^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Over $\mathbb{Z}_2$, the row reduction breaks down at the second step, producing a row of zeros, so B is not invertible. Over $\mathbb{Z}_3$, the matrix simplifies to $B^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

(c) Working over $\mathbb{R}$:

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

giving

$$C^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The calculation works over $\mathbb{Z}_2$, yielding $C^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and over $\mathbb{Z}_3$, yielding

$$C^{-1} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. (a) Expanding along the first row gives

$$\begin{aligned} \det M &= \begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = 2\begin{vmatrix} 3 & 4 \\ -2 & 8 \end{vmatrix} - (-3)\begin{vmatrix} -1 & 4 \\ -7 & 8 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ -7 & -2 \end{vmatrix} \\ &= 2(24+8) + 3(-8+28) - 2(2+21) = 2(32) + 3(20) - 2(23) = 78\,. \end{aligned}$$

(b) Expanding along the second row gives

$$\begin{aligned} \det M &= \begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = -(-1)\begin{vmatrix} -3 & -2 \\ -2 & 8 \end{vmatrix} + 3\begin{vmatrix} 2 & -2 \\ -7 & 8 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ -7 & -2 \end{vmatrix} \\ &= (-24-4) + 3(16-14) - 4(-4-21) = -28 + 3(2) - 4(-25) = 78\,. \end{aligned}$$

(c) Expanding down the third column gives

$$\begin{aligned} \det M &= \begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = -2\begin{vmatrix} -1 & 3 \\ -7 & -2 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ -7 & -2 \end{vmatrix} + 8\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} \\ &= -2(2+21) - 4(-4-21) + 8(6-3) = -2(23) - 4(-25) + 8(3) = 78\,. \end{aligned}$$

A faster calculation, for example, would be to use the first column to create zeros in the second row, expanding along which needs only a single 2×2 determinant:

$$\det M = \begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 6 \\ -1 & 0 & 0 \\ -7 & -23 & -20 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ -23 & -20 \end{vmatrix} = -60 + 138 = 78\,.$$

Over $\mathbb{Z}_{11}$, we have $\det M = 1$, which is nonzero, so M is invertible. However, over $\mathbb{Z}_{13}$, we have $\det M = 0$, so M is not invertible.

4. We have $\alpha = (1\,2\,3\,4\,5\,6) = (1\,2)(1\,3)(1\,4)(1\,5)(1\,6)$, as a product of 5 transpositions, so α is odd, and $\alpha^{-1} = (6\,5\,4\,3\,2\,1)$.

We have $\beta = (1\;2)(3\;4)(5\;6\;7\;8) = (1\;2)(3\;4)(5\;6)(5\;7)(5\;8)$, as a product of 5 transpositions, so β is odd, and $\beta^{-1} = (1\,2)(3\,4)(8\,7\,6\,5)$.

We have $\gamma = (1\,3\,5)(2\,4\,6) = (1\,3)(1\,5)(2\,4)(2\,6)$, as a product of 4 transpositions, so γ is even, and $\gamma^{-1} = (5\,3\,1)(6\,4\,2)$.

5. (a) $(6\ 5\ 4\ 3\ 2\ 1)$ (b) $(2\ 1)(3\ 6)(4\ 5)$ (c) $(9\ 1)(2\ 3\ 4\ 5)(6\ 8\ 7)$ (d) $(5\ 7\ 3\ 4\ 6\ 2)(1\ 9)$

6. We have

$$\alpha = (1\ 2)(1\ 4) = (1\ 2\ 4)$$

which is even;

$$\beta = (1\ 2\ 3)(1\ 4\ 5) = (1\ 2\ 3\ 4\ 5)$$

which is even;

$$\gamma = (1\ 2\ 3\ 4)(4\ 5\ 6)(2\ 4) = (1\ 4)(2\ 3\ 5\ 6)$$

which is even;

$$\delta = (1\ 2\ 3\ 4\ 5)(1\ 3\ 5)(2\ 5\ 6\ 7) = (1\ 5\ 3\ 4)(2\ 6\ 7)$$

which is odd;

$$\varepsilon = (1\ 2)(1\ 3)(1\ 4\ 5)(4\ 2)(3\ 5\ 7) = (1\ 4\ 7\ 3\ 2\ 5)$$

which is odd.

7. (a) The permutation corresponding to this configuration is $(1\ 2)(4\ 5\ 6\ 8)$, which is even, so the configuration is possible.

(b) The permutation corresponding to this configuration is $(4\ 7)(5\ 8\ 6)$, which is odd, so the configuration is not possible.

(c) The permutation corresponding to this configuration is $(1\ 2\ 3\ 4\ 7)$, which is even, so the configuration is possible.

(d) This configuration is possible if and only if the following configuration is possible, by an obvious shifting of squares:

1	4	7
3	2	8
6	5	

This has the corresponding permutation $(2\ 5\ 8\ 6\ 7\ 3\ 4)$, which is even, so both it and the original configuration are possible.

(e) This configuration is possible if and only if the following configuration is possible, by an obvious shifting of one square:

3	1	4
2	8	7
6	5	

This has the corresponding permutation $(1\ 2\ 4\ 3)(5\ 8)(6\ 7)$, which is odd, so both it and the original configuration are not possible.

(f) This configuration is possible if and only if the following configuration is possible, by an obvious shifting of squares:

8	7	6
1	5	4
2	3	

This has the corresponding permutation $(1\ 4\ 6\ 3\ 8)(2\ 7)$, which is odd, so both it and the original configuration are not possible.

8. Row reducing over $\mathbb{Z}_{11}$ gives

$$\left[\begin{array}{rrr|rrr} 2 & -3 & -2 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ -7 & -2 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & -3 & -4 & 0 & -1 & 0 \\ 0 & 3 & 6 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 & -7 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 7 & -1 \\ 0 & 3 & 6 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 7 & -1 \\ 0 & 0 & 1 & 1 & 3 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & -5 & -6 \\ 0 & 1 & 0 & 2 & 2 & 5 \\ 0 & 0 & 1 & 1 & 3 & 3 \end{array}\right]$$

so that

$$M^{-1} = \begin{bmatrix} -1 & -5 & -6 \\ 2 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 5 \\ 2 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix}.$$

9. (a) We have

$$\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 3 \\ 4 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 14 \\ 0 & 1 & -3 \end{vmatrix} = -14 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -14\,.$$

(b) We have

$$\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = 0\,,$$

regardless of the field so that matrix $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$ is never invertible.

(c) We have

$$\begin{vmatrix} 2 & 3 & 6 & 2 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 5 \\ 2 & 0 & -1 & -1 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 5 \\ 2 & -1 & -1 \\ 4 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 2 & -1 & 9 \\ 4 & 1 & 23 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 \\ 6 & 0 & 32 \\ 4 & 1 & 23 \end{vmatrix} = -\begin{vmatrix} 0 & 32 \\ 1 & 23 \end{vmatrix} = -(0-32) = 32\,,$$

which is nonzero in $\mathbb{R}$, and also nonzero when evaluated in $\mathbb{Z}_p$ for odd primes p, since $32 = 2^5$ is not divisible by p. Hence $\begin{bmatrix} 2 & 3 & 6 & 2 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{bmatrix}$ is invertible over $\mathbb{R}$ and over $\mathbb{Z}_p$ where p is an odd prime.

10. If a square matrix A has two identical rows or two identical columns then its determinant is unchanged when subtracting one of these rows or columns from its other identical row or column, producing another matrix B with a row or column of zeros, so that $\det A = \det B = 0$, by expanding along that row or down that column of zeros.

11. (a) Suppose that $A = A^2$ is idempotent. Put $\lambda = \det A$. Then, by the multiplicative property of determinants,

$$\lambda \;=\; \det A \;=\; \det(A^2) \;=\; (\det A)(\det A) \;=\; \lambda^2 \;,$$

so that $0 = \lambda^2 - \lambda = \lambda(\lambda - 1)$, which has roots $\lambda = 0$ or $\lambda = 1$ only. Thus the determinant of A can only be 0 or 1.

(b) Suppose that A is nilpotent, so that $A^k = 0$, the zero matrix, for some positive integer k. By the multiplicative property again,

$$0 \;=\; \det 0 \;=\; \det(A^k) \;=\; (\det A)^k \;,$$

which is only possible in field arithmetic if $\det A = 0$.

12. If α is the identity permutation then certainly $\alpha^{-1} = \alpha = \alpha^1$. Suppose α is not the identity permutation. Then $\alpha = \alpha_1\alpha_2\ldots\alpha_\ell$ where $\alpha_1, \ldots, \alpha_\ell$ are disjoint cycles of length $n_1, \ldots, n_\ell$ respectively, and $n_1 > 1$. Then

$$\alpha^{n_1\ldots n_\ell} \;=\; 1 \;,$$

the identity permutation, so that $\alpha^{-1} = \alpha^k$ where $k = (n_1 \ldots n_\ell) - 1$ is a positive integer (since $n_1 > 1$).

The transposition $\sigma = (1\ 2)$ is odd. If the set of all odd permutations formed a permutation group then in particular $\sigma^2 = 1$ would be odd. But 1 is even, not odd, which is a contradiction. Hence the odd permutations do not form a permutation group.

13. There are twelve even permutations of $\{1, 2, 3, 4\}$:

$$1\,(\text{the identity permutation})\,,\ (1\ 2)(3\ 4)\,,\ (1\ 3)(2\ 4)\,,\ (1\ 4)(2\ 3)\,,\ (1\ 2\ 3)\,,$$

$$(1\ 2\ 4)\,,\ (1\ 3\ 4)\ \ (2\ 3\ 4)\,,(1\ 3\ 2)\,,\ (1\ 4\ 2)\,,\ (1\ 4\ 3)\,,(2\ 4\ 3)\;.$$

14. Observe that A_4 is a subgroup (the whole group in fact) of size 12, the trivial subgroup $\{1\}$ has size 1, the subgroup $\{1\,,\ (1\ 2)(3\ 4)\}$ has size 2, the subgroup $\{1\,,\ (1\ 2\ 3)\,,\ (3\ 2\ 1)\}$ has size 3, and

$$F \;=\; \{\, 1\,,\ (1\ 2)(3\ 4)\,,\ (1\ 3)(2\ 4)\,,\ (1\ 4)(2\ 3)\,\}$$

is a subgroup of size 4. We prove that A_4 contains no subgroup of size 6 by contradiction. Suppose that a subgroup H exists of size 6. From the list in the previous exercise, it is unavoidable that H contains a 3-cycle. Without loss of generality (by renaming the letters if necessary) we may suppose H contains $\alpha = (1\ 2\ 3)$ and therefore also $\alpha^2 = \alpha^{-1} = (3\ 2\ 1)$. The argument splits into two cases:

Case (i): Suppose $H \cap F \neq \{1\}$.

If $\beta = (1\ 2)(3\ 4) \in H \cap F$ then

$$\alpha^{-1}\beta\alpha \;=\; (2\ 3)(1\ 4) \;\in\; H \quad \text{and} \quad \alpha\beta\alpha^{-1} \;=\; (3\ 1)(2\ 4) \;\in\; H \;,$$

from which it follows immediately that $F \subset H$. But H has exactly six elements, so

$$H = F \cup \{\alpha, \alpha^{-1}\} .$$

Similarly $H = F \cup \{\alpha, \alpha^{-1}\}$ if $(1\ 3)(2\ 4)$ or $(1\ 4)(2\ 3)$ lie in $H \cap F$. But $F \cup \{\alpha, \alpha^{-1}\}$ is not closed under composition, since

$$(1\ 2)(3\ 4)(1\ 2\ 3) = (1\ 3\ 4) \notin F \cup \{\alpha, \alpha^{-1}\} ,$$

contradicting that H is a subgroup.

Case (ii): Suppose $H \cap F = \{1\}$.

Then, from the list of elements in the previous exercise, at least one of $(1\ 2\ 4)$, $(1\ 3\ 4)$, $(2\ 3\ 4)$, $(1\ 4\ 2)$, $(1\ 4\ 3)$ or $(2\ 4\ 3)$ lies in H. But, products of these with $(1\ 2\ 3)$ or $(1\ 3\ 2)$ produce a nontrivial element of F:

$$(1\ 2\ 4)(1\ 2\ 3) = (1\ 3)(2\ 4)\,, \quad (1\ 3\ 4)(1\ 3\ 2) = (1\ 2)(3\ 4)\,, \quad (2\ 3\ 4)(1\ 2\ 3) = (2\ 1)(3\ 4)\,,$$

$$(1\ 4\ 2)(1\ 3\ 2) = (1\ 4)(2\ 3)\,, \quad (1\ 4\ 3)(1\ 2\ 3) = (1\ 4)(2\ 3)\,, \quad (2\ 4\ 3)(1\ 3\ 2) = (2\ 4)(1\ 3)\,.$$

contradicting that $H \cap F = \{1\}$.

15. (a) If A is a square matrix then $A = IAI = I^{-1}AI$, verifying that A is conjugate to itself. Thus conjugacy is reflexive.

If A and B are conjugate then $B = P^{-1}AP$ for some invertible P, so that

$$A = PBP^{-1} = (P^{-1})^{-1}BP^{-1} ,$$

verifying that A is conjugate to B. Thus conjugacy is symmetric.

Suppose that A and B are conjugate and B and C are conjugate, so $B = P^{-1}AP$ and $C = Q^{-1}BQ$ for some invertible matrices P and Q. Then PQ is invertible and $(PQ)^{-1} = Q^{-1}P^{-1}$ so that

$$(PQ)^{-1}A(PQ) = Q^{-1}(P^{-1}AP)Q = Q^{-1}BQ = C ,$$

verifying that A and C are conjugate. Thus conjugacy is transitive.

(b) Suppose A and B are conjugate, so $B = P^{-1}AP$ for some invertible matrix P. Note that $\det P \neq 0$, since P is invertible. But $PB = AP$, so that, by the multiplicative property of determinants,

$$(\det P)(\det B) = \det(PB) = \det(AP) = (\det A)(\det P) ,$$

so that, dividing through by $\det P$, we have $\det B = \det A$, as required.

(c) Suppose that A and B are conjugate, so $B = P^{-1}AP$ for some invertible matrix P. Let $\lambda \in F$. Then

$$P^{-1}(A - \lambda I)P = P^{-1}AP - P^{-1}(\lambda I)P = B - \lambda(P^{-1}P) = B - \lambda I .$$

This verifies that $A - \lambda I$ and $B - \lambda I$ are also conjugate. By the previous part

$$\det(A - \lambda I) = \det(B - \lambda I) .$$

In particular $\det(A - \lambda I) = 0$ precisely when $\det(B - \lambda I) = 0$.

16. Put $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the lines $ax + by = k$ and $cs + dy = \ell$ have a unique intersection point (x, y) if and only if the matrix equation

$$M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ \ell \end{bmatrix}$$

has a unique solution. If $0 \neq ad - bd = \det M$ then M is invertible so that

$$\begin{bmatrix} x \\ y \end{bmatrix} = M^{-1} \begin{bmatrix} k \\ \ell \end{bmatrix}$$

is the unique solution. Conversely, if the matrix equation has a unique solution then the reduced echelon form of M must be the identity matrix I, so that M is invertible, so that $ad - bc = \det M \neq 0$. This proves the lines have a unique point of intersection if and only if $ad - bc \neq 0$.