The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Exercises for Week 5 (beginning 1 April)** 2019

Important Ideas and Useful Facts:

(i) Eigenvalues and eigenvectors: Let M be a square matrix, $\mathbf{x}$ a nonzero column vector and λ a scalar such that

$$M\mathbf{x} \;=\; \lambda\mathbf{x}\,.$$

Then λ is called an *eigenvalue* of M and $\mathbf{x}$ is called an *eigenvector* of M associated with or corresponding to the eigenvalue λ.

(ii) The eigenspace of a matrix: The *eigenspace* of a square matrix M associated with an eigenvalue λ is the collection

$$\left\{ \mathbf{v} \;\middle|\; M\mathbf{v} = \lambda\mathbf{v} \right\} \;=\; \left\{ \mathbf{v} \;\middle|\; (\lambda I - M)\mathbf{v} = \mathbf{0} \right\}$$

comprising all of the eigenvectors of M associated with λ and the zero vector (which is never an eigenvector).

(iii) Description of eigenvalues in terms of determinants: A scalar λ is an eigenvalue of a square matrix M if and only if

$$\det(\lambda I - M) \;=\; 0\,.$$

(iv) The characteristic polynomial of a square matrix: The expression $\det(\lambda I - M)$ is always a polynomial in λ and is called the *characteristic polynomial* of M. Thus the eigenvalues of a square matrix are precisely the roots of its characteristic polynomial.

(v) Finding eigenspaces: Finding the eigenspace corresponding to the eigenvalue λ of a matrix M is equivalent to solving the homogeneous system with coefficient matrix $\lambda I - M$. After the eigenspace has been found, substituting particular values of the parameters yields particular eigenvectors.

(vi) Eigenvalues of a triangular matrix: The eigenvalues of a triangular matrix are simply the diagonal entries.

(vii) The Cayley-Hamilton Theorem: Every square matrix M is the root of its own characteristic polynomial, that is,

$$\chi(M) \;=\; 0\,,$$

where $\chi(\lambda) = \det(\lambda I - M)$ denotes the characteristic polynomial, $\chi(M)$ is the result of evaluating the matrix expression obtained from $\chi(\lambda)$ by substituting M for the indeterminate λ and I for the constant 1, and 0 denotes the zero matrix.

(viii) Reflection matrices: A *reflection matrix* in the real plane has the form

$$M = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

for some real θ, and corresponds to reflection in the plane through a line through the origin making an angle θ with the positive x-axis. The eigenvalues of M are ± 1. The eigenspace corresponding to 1 is the line of reflection. The eigenspace corresponding to -1 is the line through the origin perpendicular to the line of reflection.

(ix) Rotation matrices: A *rotation matrix* in the real plane has the form

$$M = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

for some real θ, and corresponds to rotation in the plane anticlockwise through an angle θ about the origin. The eigenvalues of M are the complex numbers

$$e^{i\theta} = \cos\theta \pm i \sin\theta$$

where $i = \sqrt{-1}$. The eigenvalues are real if and only if θ is an integer multiple of π, in which case all nonzero vectors are eigenvectors, corresponding to eigenvalue 1 if the integer multiple is even, and corresponding to -1 if the integer multiple is odd.

(x) The orthogonal group of the real plane: Let G be the set of all reflection and rotation matrices in the plane. Then G forms a group with respect to matrix multiplication, called the *orthogonal group* (which generalises to higher dimensions taking all square matrices of fixed size whose rows, equivalently columns, form an orthonormal set).

(xi) The circle group: The set of all complex numbers of magnitude 1 forms a group G under multiplication, called the *circle group*. Then G is the same group (up to isomorphism) as the group of all rotations of the plane (which is a subgroup of index two in the orthogonal group).

(xii) The finite dihedral groups: Let G be the group of symmetries of the regular n-gon, where $n \geq 3$, under composition of symmetries. Then G contains exactly n rotations and n reflections and can be described completely as

$$G = \{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}, \beta, \alpha\beta, \ldots, \alpha^{n-1}\beta\}$$

where α is a rotation of the polygon about its centre by $2\pi/n$ radians and β is a reflection about any fixed axis of reflectional symmetry (in which case $\beta, \alpha\beta, \ldots, \alpha^{n-1}\beta$ becomes an exhaustive list of reflections about all axes of symmetry).

(xiii) The infinite dihedral group: Let G be the group of symmetries of the circle under composition of symmetries. Then G is the same group (up to isomorphism) as the orthogonal group of the plane. If $g, h \in G$ where g is a rotation of the circle by θ radians and h is a reflection of the circle about a fixed axis $\mathcal{L}$ of reflectional symmetry, then $h^2 = 1$, the identity mapping, $hgh = g^{-1}$, the rotation of the circle $-\theta$ radians, and

$$g^{-1}hg = g^{-2}h = hg^2$$

is a reflection of the circle about the axis of reflectional symmetry formed by rotating $\mathcal{L}$ by θ radians.

Questions labelled with an asterisk are suitable for students aiming for a distinction or higher.

Tutorial Exercises:

1. Find eigenvalues and eigenspaces for the following matrices, working over $\mathbb{R}$:

(a) $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

2. Find eigenvalues and eigenspaces for the following matrices, working over $\mathbb{R}$:

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (c)* $C = \begin{bmatrix} -7 & -2 & 6 \\ -2 & 1 & 2 \\ -10 & -2 & 9 \end{bmatrix}$

3. Let A be a square matrix over a field F with eigenvalue λ. Prove the following implications:

(a) $A^2 = 0 \Rightarrow \lambda = 0$ (b) $A^2 = A \Rightarrow (\lambda = 0 \text{ or } \lambda = 1)$

(c) $A^2 = I \Rightarrow (\lambda = 1 \text{ or } \lambda = -1)$

4. Consider the group G of symmetries of a regular hexagon, generated by a rotation α one sixth of a turn and a reflection β.

(a) Write out all of the elements of G as simply as possible. What happens if you reverse the order of the generators in each of the expressions that you found?

(b) Evaluate $\beta\alpha^5\beta^5\alpha^{-3}\beta^{-3}\alpha^8\beta$ as simply as possible.

(c)* Find a rotation matrix A and a reflection matrix B that together generate a group H in a one-one correspondence to G. Explain why matrix multiplication in H corresponds to composition of symmetries in G under this correspondence (thus exhibiting an *isomorphism* between H and G).

5.* Consider the permutations $\alpha = (1\ 2\ 3\ 4\ 5\ 6)$ and $\beta = (1\ 6)(2\ 5)(3\ 4)$. Verify that $\alpha^6 = \beta^2 = 1$ and $\beta^{-1}\alpha\beta = \alpha^{-1}$. Write out all permutations in $\langle \alpha, \beta \rangle$ using cycle notation. Find all $\gamma \in \mathrm{Sym}(6)$ such that $\gamma^{-1}\alpha\gamma = \alpha^{-1}$. What do you notice?

6.* Consider the matrix $M = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -3 \end{bmatrix}$.

(a) Verify directly that $M^2 = 3M - 2I$ and that

$$\chi(\lambda) = \det(\lambda I - M) = (\lambda - 1)^2(\lambda - 2) = \lambda^3 - 4\lambda^2 + 5\lambda - 2\,.$$

(b) What conclusion follows from the second part of (a) and the Cayley-Hamilton Theorem? How is this conclusion reconciled with the first part of (a)?

(c) Prove by induction that, for all positive integers k,

$$M^k = (2^k - 1)M + (2 - 2^k)I\,.$$

Now deduce this formula for all integers k.

(d) Evaluate M^5, M^{-1} and M^{-5}.

Further Exercises:

7. Consider the matrices

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad B = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}, \quad C = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

for some $\theta, \phi \in \mathbb{R}$. Describe each of the following as simply as possible:

(a) A^2 (b) B^2 (c) A^{-3} (d) AB (e) AC

(f) BA (g) BC (h) ABA (i) BA^2B (j) BAC

8. Suppose that $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Verify that the characteristic polynomial of M is

$$\chi(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc\,.$$

Now also verify that

$$M^2 - (a+d)M + (ad-bc)I = 0\,.$$

This verifies the Cayley-Hamilton Theorem directly for 2×2 matrices.

9. Let λ be an eigenvalue of a matrix M.

(a) Show that λ^k is an eigenvalue of M^k for all positive integers k.

(b) Show that if M is invertible then λ is nonzero and λ^{-1} is an eigenvalue of M^{-1}.

10.* Working over $\mathbb{C}$, find the eigenspaces of a rotation matrix.

11.* Let A be a square matrix. Denote by A_{ij} the square matrix that results by deleting the ith row and jth column of A. Define the *adjugate matrix* adj$A$ to be the square matrix, of the same size as A, whose (j,i)-entry is $(-1)^{i+j}\det A_{ij}$. Quoting results about determinants, verify that

$$A(\mathrm{adj}A) = (\mathrm{adj}A)A = (\det A)I\,.$$

This proves that A is invertible if and only if $\det A \neq 0$, in which case $A^{-1} = \frac{1}{\det A}\mathrm{adj}A$.

12.* Use the adjugate matrix to verify the following fact, known as *Cramer's Rule*: if M is an invertible $n \times n$ matrix and $\mathbf{c}$ a column vector, then the equation $M\mathbf{x} = \mathbf{c}$ has a unique solution $\mathbf{x}$ whose ith entry is

$$x_i = \frac{\det M_i}{\det M}$$

where M_i is the matrix obtained by replacing the ith column of M by $\mathbf{c}$. Use Cramer's Rule to solve the following system of equations:

$$\begin{aligned} 2x + 3y + 4z &= -4 \\ 5x + 5y + 6z &= -3 \\ 3x + y + 2z &= -1 \end{aligned}$$