THE UNIVERSITY OF SYDNEY

MATH2022 LINEAR AND ABSTRACT ALGEBRA

Semester 1 **Week 5 Hints and Short Solutions** 2019

**1.** (a) $2, \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| t \in \mathbb{R} \right\}, 1, \left\{ \begin{bmatrix} 0 \\ t \end{bmatrix} \middle| t \in \mathbb{R} \right\}$. (b) $2, \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| t \in \mathbb{R} \right\}, 3, \left\{ \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} \middle| t \in \mathbb{R} \right\}$.
(c) $2, \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| t \in \mathbb{R} \right\}$.

**2.** (a) $1, \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \middle| t \in \mathbb{R} \right\}, 2, \left\{ \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \middle| t \in \mathbb{R} \right\}, 3, \left\{ \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} \middle| t \in \mathbb{R} \right\}$.
(b) $2, \left\{ \begin{bmatrix} \frac{t}{2} \\ t \\ s \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}, 1, \left\{ \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$.
(c) $3, \left\{ \begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} \middle| t \in \mathbb{R} \right\}, 1, \left\{ \begin{bmatrix} t \\ -t \\ t \end{bmatrix} \middle| t \in \mathbb{R} \right\}, -1, \left\{ \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} \middle| t \in \mathbb{R} \right\}$.

**3.** We have $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector $\mathbf{v}$. Manipulate $A^2\mathbf{v}$ in each case to get an equation in $F$ involving $\lambda$.

**4.** (a) $G = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta, \alpha^4\beta, \alpha^5\beta\}$. (b) $\alpha^2$.
(c) Let $A$ be a rotation matrix using $\pi/3$ radians, and $B$ any reflection matrix.

**5.** $\langle \alpha, \beta \rangle = \{1, (1\ 2\ 3\ 4\ 5\ 6), (1\ 3\ 5)(2\ 4\ 6), (1\ 4)(2\ 5)(3\ 6), (1\ 5\ 3)(2\ 6\ 4), (1\ 6\ 5\ 4\ 3\ 2), (1\ 6)(2\ 5)(3\ 4), (1\ 5)(2\ 4), (1\ 4)(2\ 3)(5\ 6), (1\ 3)(4\ 6), (1\ 2)(3\ 6)(4\ 5), (2\ 6)(3\ 5)\}$, $\gamma \in \{\beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta, \alpha^4\beta, \alpha^5\beta\}$.

**6.** $M^5 = \begin{bmatrix} 32 & 62 & 124 \\ 62 & 125 & 248 \\ -31 & -62 & -123 \end{bmatrix}, \ M^{-1} = \begin{bmatrix} \frac{1}{2} & -1 & -2 \\ -1 & -1 & -4 \\ \frac{1}{2} & 1 & 3 \end{bmatrix}, \ M^{-5} = \frac{1}{32} \begin{bmatrix} 1 & -62 & -124 \\ -62 & -92 & -248 \\ 31 & 62 & 156 \end{bmatrix}$.

**7.** (a) rotation $2\theta$. (b) rotation 0. (c) rotation $-3\theta$. (d) reflection $\frac{\theta+2\phi}{2}$. (e) reflection $\frac{3\theta}{2}$.
(f) reflection $\frac{2\phi-\theta}{2}$. (g) rotation $2(\phi-\theta)$. (h) $B$. (i) rotation $-2\theta$. (j) rotation $2\phi-3\theta$.

**8.** Straightforward expansions and simplifications.

**9.** For part (a), iterate multiplying the eigenvector by the matrix $M$. For part (b), if $\lambda = 0$, use invertibility of $M$ to contradict that an eigenvector is nonzero.

**10.** $\left\{ \begin{bmatrix} iz \\ z \end{bmatrix} \middle| z \in \mathbb{C} \right\}$ for $\cos\theta + i\sin\theta$, $\left\{ \begin{bmatrix} -iz \\ z \end{bmatrix} \middle| z \in \mathbb{C} \right\}$ for $\cos\theta - i\sin\theta$.

**11.** Use determinant expansions along rows and the fact that a matrix with identical rows has zero determinant.

**12.** Use determinant expansions down columns and compare coefficients on both sides of the matrix equation $\mathbf{x} = M^{-1}\mathbf{c}$. System has solution $(x, y, z) = (1, 2, -3)$.