The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Exercises for Week 7 (beginning 8 April)** 2019

Important Ideas and Useful Facts:

(i) Cartesian products of sets and inherited coordinatewise operations: If $A_1, A_2, \ldots, A_n$ are sets then we may form the Cartesian product

$$A_1 \times A_2 \times \ldots \times A_n \;=\; \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

which has coordinate-wise operations inherited from $A_1, \ldots, A_n$, in the case that these have arithmetic operations of the same type (such as addition or multiplication). In particular, if F is a field and $n \geq 1$ then we may form the *Cartesian power*

$$F^n \;=\; A_1 \times \ldots \times A_n \;=\; \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in F\}$$

where $A_1 = \ldots = A_n = F$, with coordinatewise addition, multiplication and scalar multiplication. In this case the mapping

$$(a_1, \ldots, a_n) \mapsto \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

is a bijection between F^n and the set V of all column vectors over F of length n preserving addition and scalar multiplication. Both F^n and V are examples of *vector spaces* (see later), and the previous statement says that F^n and V are *vector space isomorphic*. We also define $F^0 = \{0\}$, called the *trivial vector space*.

(ii) Identification of n-tuples with row vectors: It is common to make the following identification of an n-tuple with the row vector obtained by deleting commas and replacing round brackets with square brackets:

$$(a_1, a_2, \ldots, a_n) \;\equiv\; [\, a_1 \; a_2 \; \ldots \; a_n \,] \,,$$

and then coordinatewise addition and scalar multiplication of n-tuples becomes addition and scalar multiplication of row vectors as $1 \times n$ matrices.

(iii) Linear transformations (special case): The Cartesian power F^n is the prototype structure for an *n-dimensional vector space* (see later for definitions). A function $L : F^m \to F^n$, where m and n are nonnegative integers, is called a *linear transformation* if L respects coordinatewise addition and scalar multiplication, that is, for all $\mathbf{v}, \mathbf{w} \in F^m$ and $\lambda \in F$,

$$L(\mathbf{v} + \mathbf{w}) \;=\; L(\mathbf{v}) + L(\mathbf{w}) \quad \text{and} \quad L(\lambda \mathbf{v}) \;=\; \lambda L(\mathbf{v}) \;;$$

equivalently, for all $\mathbf{v}, \mathbf{w} \in F^m$ and $\lambda, \mu \in F$,

$$L(\lambda \mathbf{v} + \mu \mathbf{w}) \;=\; \lambda L(\mathbf{v}) + \mu L(\mathbf{w}) \;,$$

and we say that L *respects* or *preserves linear combinations*. If $m = n$ then L is called a *linear operator*. It is traditional to use functional notation for linear transformations and operators and to compose them in the reverse order to which they are written down (from right to left, by contrast with the left to right convention commonly used by algebraists). The composite of linear transformations, when defined, is also a linear transformation.

(iv) Standard basis: For $1 \le i \le n$, let $\mathbf{e}_i$ be the n-tuple with 0 in each place except for 1 in the ith place. Put $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, called the *standard basis* for F^n. If $\mathbf{v} = (v_1, \ldots, v_n) \in F^n$ then

$$\mathbf{v} \;=\; v_1\mathbf{e}_1 + \ldots + v_n\mathbf{e}_n \;,$$

so that B has the so-called *spanning property*. Also B is *linearly independent* (see later).

(v) Matrix of a linear transformation: Let $L : F^m \to F^n$ be a linear transformation and let B_m be the standard basis for F^m. Form the $n \times m$ matrix M_L where, for $1 \le i \le m$, the ith column of M_L is $\big(L(\mathbf{e}_i)\big)^\top$, the transpose of the row vector $L(\mathbf{e}_i)$. The action of L on row vectors corresponds to matrix multiplication of column vectors by M_L in the following sense:

$$L(\mathbf{v}) \;=\; \mathbf{w} \quad \text{if and only if} \quad M_L\mathbf{v}^\top \;=\; \mathbf{w}^\top \;.$$

(vi) Matrix multiplication corresponds to composition of linear transformations: Let $L_1 : F^m \to F^n$ and $L_2 : F^n \to F^q$ be linear transformations, so that $L_2L_1 = L_2 \circ L_1 : F^m \to F^q$. Then

$$M_{L_2L_1} \;=\; M_{L_2}M_{L_1} \;.$$

(viii) Isomorphisms of groups: If G and H are groups then a bijection $\phi : G \to H$ is called an *isomorphism* if ϕ preserves the group operation, that is, $(g_1g_2)\phi = (g_1\phi)(g_2\phi)$ for all $g_1, g_2 \in G$. If there exists an isomorphism between two groups G and H then we say that G and H are *isomorphic* and write $G \cong H$. Then $\cong$ is an equivalence relation on the class of groups, that is, $\cong$ is reflexive, symmetric and transitive.

(xi) Cyclic groups: A group G is called *cyclic* if there exists an element g, called the *generator* of G such that every element is a power (possibly negative) of g (expressing the group operation multiplicatively). If a cyclic group is finite then every element is a positive power of the chosen generator. Two cyclic groups are isomorphic if and only if they have the same number of elements, and all isomorphisms are induced by mappings that take a generator of one group to a generator of the other. Expressed additively, every cyclic group is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_n$ for some positive integer n.

(ix) Shear matrices: A *(standard) shear matrix* is an elementary matrix of the form

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

for some real number k, which corresponds to the shear transformation of the xy-plane that fixes the x-axis and shifts points sideways proportional to their y-coordinates (with factor k of proportionality). The only eigenvalue of a shear matrix is 1. When $k \neq 0$, a shear matrix is not diagonalisable and the eigenspace corresponding to 1 may be identified with the x-axis.

(x) Invertible linear transformations of the plane: Denote by $\mathcal{L}$ the group of invertible linear transformations of the plane under composition, and by $G = \mathrm{GL}_2(\mathbb{R})$ the group of invertible real 2×2 matrices under matrix multiplication. Then $\mathcal{L}$ and G are isomorphic under the mapping that associates a matrix with a linear transformation with respect to the standard basis. Every invertible linear transformation of the plane decomposes as the composite of a shear, a pair of dilations in the x and y-directions respectively and either a rotation (orientation preserving) or a reflection (orientation reversing). Equivalently, every invertible 2×2 real matrix decomposes as a product of a shear matrix, a diagonal matrix and either a rotation or a reflection matrix.

Tutorial Exercises:

1. (a) Write out all of the elements (ordered pairs) of the sets $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_3^2$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$.

(b) Fill out addition tables for $\mathbb{Z}_2^2$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$.

(c) Show that $\mathbb{Z}_2 \times \mathbb{Z}_3$ can be generated by a single element under addition, that is, we can get to every element by adding that single element to itself enough times.

(d) Explain why each of $\mathbb{Z}_2^2$ and $\mathbb{Z}_3^2$ can be generated by two elements under addition but not by a single element.

(e)* Explain why $\mathbb{Z}_2^3$ can be generated by three elements under addition, but not by fewer than three elements.

2. Let F be a field, m, n be nonnegative integers and $L : F^m \to F^n$ be a mapping. Verify that L respects addition and scalar multiplication if and only if L respects linear combinations.

3. Let $L : F^2 \to F^2$ be given by the rule

$$L(x, y) \;=\; (ax + by, cx + dy)$$

where a, b, c, d are fixed constants from field F. Verify that L is a linear transformation. Write down the matrix M such that $M\mathbf{v}^\top = \mathbf{w}^\top$ whenever $L(\mathbf{v}) = \mathbf{w}$.

4. Consider the linear transformations $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^4$ defined by the following rules:

$$f(x, y) \;=\; (x, x + y, 3x - 2y)\,, \qquad g(x, y, z) \;=\; (x + 2y, 2y - z, x - 3y, x + y + z)\,.$$

(a) Verify directly that f is indeed a linear transformation.

(b) Find the matrices M_f and M_g that give rise to f and g respectively.

(c) Find directly the rule for the composite linear transformation $gf = g \circ f : \mathbb{R}^2 \to \mathbb{R}^4$, find its matrix M_{gf} and check that $M_{gf} = M_g M_f$.

5. (a) Draw the following parallelograms in the xy-plane, each given by four vertices that move anticlockwise around the boundary, anchored at the origin:

(i) $\mathcal{P}_1 : O(0, 0)\,,\ A_1(2, 1)\,,\ B_1(3, 3)\,,\ C_1(1, 2)\,;$

(ii) $\mathcal{P}_2 : O(0, 0)\,,\ A_2(\sqrt{5}, 0)\,,\ B_2\big(\frac{9}{\sqrt{5}}, \frac{3}{\sqrt{5}}\big)\,,\ C_2\big(\frac{4}{\sqrt{5}}, \frac{3}{\sqrt{5}}\big)\,;$

(iii) $\mathcal{P}_3 : O(0, 0)\,,\ A_3(1, 0)\,,\ B_3\big(\frac{9}{5}, 1\big)\,,\ C_3\big(\frac{4}{5}, 1\big)\,;$

(iv) $\mathcal{P}_4 : O(0, 0)\,,\ A_4(1, 0)\,,\ B_4(1, 1)\,,\ C_4(0, 1)\,;$

Describe simple geometric transformations of the xy-plane that transform $\mathcal{P}_1$ to $\mathcal{P}_2$, $\mathcal{P}_2$ to $\mathcal{P}_3$, and $\mathcal{P}_3$ to $\mathcal{P}_4$ respectively.

(b) Use part (a) to find a rotation matrix R, a diagonal matrix D and a shear matrix S such that

$$SDRM_1 \;=\; I\,,$$

where $M_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Now write M_1 as a product of a rotation matrix, a diagonal matrix and a shear matrix, in that order.

(c) Write the matrix M_2 as a product of a reflection matrix, a diagonal matrix and a shear matrix, in that order, where $M_2 = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$.

Further Exercises:

6. Let F be a field, m, n, q be nonnegative integers and $L_1 : F^m \to F^n$ and $L_2 : F^n \to F^q$ be linear transformations. Verify that $L_2 L_1 = L_2 \circ L_1 : F^m \to F^q$ is also a linear transformation (that is, the composite of linear transformation is a linear transformation).

7. Consider the linear transformations $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$ given by the following rules:

$$f(x,y) = (2x+y, x+2y)\,, \qquad g(x,y) = (-y,x)\,, \qquad h(x,y) = (y,x)$$

(a) Find the corresponding matrices that give rise to each of them.

(b) Find rules for gf, $g^2 f$, $g^3 f$, both directly and also using matrix multiplication.

(c) The *unit square* in the xy-plane has vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. On a single diagram, draw images of the unit square after applying f, gf, $g^2 f$ and $g^3 f$.

(d) Find rules for f^{-1}, g^{-1} and h^{-1}. Use the matrix for gf to find the rule for $(gf)^{-1}$ and then check that this coincides with the rule for the composite $f^{-1}g^{-1}$.

(e) What geometric effects occur to the images of the unit square in part (c) after applying g and h? Check that $g^4 = h^2 = \text{id}$ where id is the identity function on $\mathbb{R}^2$ and $hgh = g^{-1}$. (It follows that the functions g and h generate a group under composition isomorphic to the group of symmetries of the square.)

8. (a) Let A and B be sets. What is $A \times B$ if A or B is empty? What is the size of $A \times B$ if A and B are finite?

(b) Let A, B and C be nonempty sets. Explain briefly why

$$(A \times B) \times C \qquad \text{and} \qquad A \times B \times C$$

are in fact different sets. Exhibit a bijective mapping between them (which is in fact used by everyone implicitly to identify corresponding elements of these sets).

(c) Let A and B be different nonempty sets. Explain briefly why $A \times B$ and $B \times A$ are also different sets, but nevertheless exhibit a bijective mapping between them.

(d)* Find a nonempty set A with at least two elements such that there is a bijection between A and $A \times A$. Deduce that there is a bijection between A and A^n for $n \geq 2$.

9. Verify that if $A_1, A_2, \ldots, A_k$ are abelian groups with respect to addition, then

$$A_1 \times A_2 \times \ldots \times A_k$$

becomes an abelian group under coordinatewise addition. Thus $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k}$ is an abelian group under addition for any positive integers $n_1, n_2, \ldots, n_k$. (It is a theorem that every finite abelian group arises in this way, up to isomorphism.)

10. (a) Find two isomorphisms between the cyclic groups $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_6$ under addition. Explain briefly why there can be no others.

(b)* Find an additive generator for the group $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ and therefore deduce that $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ and $\mathbb{Z}_{30}$ are isomorphic cyclic groups. Explain why the additive groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$ and $\mathbb{Z}_{20}$ are not isomorphic.

11.* Let $\phi : G \to H$ and $\psi : H \to K$ be group isomorphisms. Verify that $\phi^{-1} : H \to G$ and $\phi\psi : G \to K$ are group isomorphisms.

12.* Let G be the subgroup of Sym(5) generated by $\alpha = (1\ 2\ 3)(4\ 5)$ and $\beta = (1\ 3)$. Write out all of the permutations in G, after explaining why $G = \{\alpha^i \beta^j \mid 0 \leq i \leq 5\,,\ 0 \leq j \leq 1\}$. Explain why G is isomorphic to the symmetry group of the hexagon.