The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 — **Week 7 Hints and Short Solutions** — 2019

1. (a) $\{(0,0),(0,1),(1,0),(1,1)\}$, $\{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$, $\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$, $\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$

(c) Try $(1,1)$ or $(1,2)$.

(d) Try $(1,0)$, $(0,1)$. One ordered pair generates too few elements.

(e) Try $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Two ordered triples generate too few elements.

2. For converse, observe that $\mathbf{v}+\mathbf{w} = 1\mathbf{v}+1\mathbf{w}$ and $\lambda\mathbf{v} = \lambda\mathbf{v}+0\mathbf{w}$ are linear combinations.

3. $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

4. (b) $M_f = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 3 & -2 \end{bmatrix}$, $M_g = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(c) $gf(x,y) = (3x+2y, -x+4y, -2x-3y, 5x-y)$, $M_{gf} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \\ -2 & -3 \\ 5 & -1 \end{bmatrix}$

5. (a) rotation $\tan^{-1} 0.5$ clockwise; dilation by $\frac{1}{\sqrt{5}}$ and by $\frac{\sqrt{5}}{3}$; shear left by $\frac{4}{5}$.

(b) $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{5} & 0 \\ 0 & \frac{3}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{bmatrix}$ (c) $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} \sqrt{5} & 0 \\ 0 & \frac{3}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{bmatrix}$

6. Use facts, in succession, that L_1 and L_2 preserve linear combinations.

7. (a) $M_f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $M_g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $M_h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $(x,y) \mapsto (-x-2y, 2x+y)$, $(-2x-y, -x-2y)$, $(x+2y, -2x-y)$

(d) $(x,y) \mapsto \frac{1}{3}(2x-y, -x+2y)$, $(y,-x)$, (y,x), $\frac{1}{3}(x+2y, -2x-y)$

(e) quarter rotation, reflection in $y = x$.

8. (a) $\emptyset$, $|A||B|$ (b) $((a,b),c) \mapsto (a,b,c)$ (c) $(a,b) \mapsto (b,a)$ (d) $\{(a_1, a_2, \ldots) \mid \text{each } a_i \in \mathbb{Z}_2\}$

9. Use respective group properties holding in each coordinate.

10. (a) Induced by $1 \mapsto (1,1)$ or $1 \mapsto (1,2)$.

(b) Try $(1,1,1)$. Any single element of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$ generates at most 10 elements.

11. For composites, mimic the proof for linear transformations.

12. 1,(1 2 3)(4 5),(1 3 2),(4 5),(1 2 3),(1 3 2)(4 5),(1 3),(1 2)(4 5),(2 3),(1 3)(4 5), (1 2),(2 3)(4 5)