THE UNIVERSITY OF SYDNEY

MATH2022 LINEAR AND ABSTRACT ALGEBRA

Semester 1 **Exercises for Week 8 (beginning 15 April)** 2019

Important Ideas and Useful Facts:

(i) Abstract vector spaces: Given a fixed field F, a *vector space over* F is an abelian group V with respect to addition, which is compatible with scalar multiplication by elements of F (denoted by juxtaposition), in the following respects:

$$(\forall \lambda, \mu \in F)(\forall \mathbf{v}, \mathbf{w} \in V) \quad (\lambda + \mu)\mathbf{v} \ = \ \lambda\mathbf{v} + \mu\mathbf{v} \quad \text{and} \quad \lambda(\mathbf{v} + \mathbf{w}) \ = \ \lambda\mathbf{v} + \lambda\mathbf{w} \ ,$$

$$(\forall \lambda, \mu \in F)(\forall \mathbf{v} \in V) \quad \lambda(\mu\mathbf{v}) \ = \ (\lambda\mu)\mathbf{v} \ ,$$

and

$$(\forall \mathbf{v} \in V) \quad 1\mathbf{v} \ = \ \mathbf{v} \ .$$

Here 1 is the multiplicative identity element of F and the addition symbol $+$ has to be read in context, belonging either to V or to F. It is an important theorem that V is isomorphic to F^n for some n (where n may be infinite, with an appropriate interpretation).

(ii) Vector space isomorphism: A mapping $\phi : V \to W$, where V and W are vector spaces over a field F is called a *vector space isomorphism* if it is a bijection that preserves addition and scalar multiplication, that is, $\phi(\mathbf{v} + \mathbf{w}) = \phi(\mathbf{v}) + \phi(\mathbf{w})$ and $\phi(\lambda\mathbf{v}) = \lambda\phi(\mathbf{v})$ for all $\mathbf{v}, \mathbf{w} \in V$ and $\lambda \in F$ (or, equivalently, preserves linear combinations).

(iii) Important examples of vector spaces: Let F be a field.

(a) The *trivial vector space* is $F^0 = \{\mathbf{0}\}$, consisting of the zero vector with trivial addition and scalar multiplication.

(b) If $n \geq 1$ then F^n, the *Cartesian power*, consisting of all n-tuples of elements of F, forms a vector space with respect to coordinate-wise addition and scalar multiplication. We may identify n-tuples with row vectors of length n, in which case the vector addition and scalar multiplication of n-tuples become addition and scalar multiplication of row matrices.

(c) If $m, n \geq 1$ then the set $\mathrm{Mat}_{m,n}$ of all $m \times n$ matrices forms a vector space with respect to matrix addition and scalar multiplication. In particular, $\mathrm{Mat}_{1,n}$, the vector space of row matrices, is identified with F^n. The vector space $\mathrm{Mat}_{m,1}$ of column matrices of length m is isomorphic to F^m under the mapping that takes a matrix to its transpose.

(d) If $n \geq 0$ then the set $\mathbb{P}_n$ of all polynomials, with coefficients from F, of degree at most n forms a vector space with respect to addition of polynomials and multiplication by constants. Then $\mathbb{P}_n$ is isomorphic to F^{n+1}.

(e) Let X be a nonempty set. Then the set of all functions from X into F, denoted by F^X, forms a vector space with respect to addition of functions and multiplication of a function by a scalar, defined by the following rules, for $f, g \in F^X$ and $\lambda \in F$:

$$(f + g)(x) \ = \ f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) \ = \ \lambda f(x) \quad \text{for all } x \in X.$$

(iv) Subspaces: A *subspace* of a vector space V over a field F is a nonempty subset S of V that is closed under vector addition and scalar multiplication, that is, for all $\mathbf{v}, \mathbf{w} \in S$ and $\lambda \in S$,

$$\mathbf{v} + \mathbf{w} \in S \quad \text{and} \quad \lambda \mathbf{v} \in S \ ,$$

or, equivalently, S is closed under taking linear combinations, that is,

$$(\forall \mathbf{v}_1, \mathbf{v}_2 \in S)(\forall \lambda_1, \lambda_2 \in F) \quad \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \in S \ .$$

A subspace S of a vector space V becomes a vector space in its own right, using the vector space operations of V restricted to S.

(v) Intersections of subspaces: Let V be a vector space. The intersection of any collection of subspaces of V is also a subspace of V. This implies that if X is any subset of V then there exists a smallest subspace of V containing X, denoted by $\langle X \rangle$, and referred to also as the *span of* X (see more below), namely

$$\langle X \rangle \ = \ \bigcap \{ S \mid S \text{ is a subspace of } V \text{ containing } X \} \ ,$$

the intersection of all subspaces of V containing X.

(vi) Linear combinations: For $k \geq 1$, a *linear combination* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ is an expression of the form

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \ldots + \lambda_k \mathbf{v}_k$$

for some scalars $\lambda_1, \ldots, \lambda_k$. If $k = 1$ then this is interpreted as a scalar multiple of $\mathbf{v_1}$. Note that since $0\mathbf{v} = \mathbf{0}$, for any vector $\mathbf{v}$, the zero vector is always a linear combination of any collection of vectors.

(vii) The span of a set of vectors: Let X be a subset of a vector space V over a field F. The *span* of X, denoted by $\langle X \rangle$ is defined to be $\{\mathbf{0}\}$, the trivial subspace of V, if $X = \emptyset$, and otherwise

$$\langle X \rangle \ = \ \{\text{all possible linear combinations of finite collections of vectors from } X\} \ .$$

It follows, in both cases, that $\langle X \rangle$ is the smallest subspace of V containing X (see above). If $X = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ then

$$\langle X \rangle \ = \ \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \ldots + \lambda_k \mathbf{v}_k \mid \lambda_1, \ldots, \lambda_k \in F\}$$

(viii) Row and column spaces of a matrix: Let M be an $m \times n$ matrix. The *row space* of M is the vector space of row vectors of length n spanned by the rows of M. The *column space* of M is the vector space of column vectors of length m spanned by the columns of M. Two matrices of the same size have the same row [column] space if and only if they are *row [column] equivalent*, that is, can be obtained from one another by elementary row [column] operations. The nonzero rows of any row echelon form for M span the row space of M (and in fact form a *basis*, see later). An analogous statement hold for the column space.

(ix) Null space of a matrix: Let M be an $m \times n$ matrix over a field F. The *null space* of M may refer either to the vector space

$$\{\text{column vectors } \mathbf{v} \text{ of length } n \mid M\mathbf{v} = \mathbf{0}\} \ ,$$

or the solution space of the associated homogeneous system of m equations in n variables:

$$\{\mathbf{v} \in F^n \mid M\mathbf{v}^\top = \mathbf{0}\} \ .$$

Tutorial Exercises:

1. Explain how the set of complex numbers

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\} ,$$

where $i = \sqrt{-1}$, becomes a vector space over the field $\mathbb{R}$. How might one identify complex numbers with geometric vectors in the plane? Find a spanning set for $\mathbb{C}$ consisting of two elements.

2. Consider the following subsets of the real vector space $\mathbb{R}^2$:

$$\begin{aligned} S_1 &= \{(x, y) \mid x + y = 0\} , & S_2 &= \{(x, y) \mid x + y = 1\} , \\ S_3 &= \{(x, y) \mid x + y \geq 0\} , & S_4 &= \{(x, y) \mid x^2 + y^2 = 1\} . \end{aligned}$$

Describe each of these sets geometrically and decide whether it is a subspace of $\mathbb{R}^2$.

3. Consider the following subsets of the real vector space $\mathbb{R}^3$:

$$\begin{aligned} S_1 &= \{(x, y, z) \mid 2x + 3y + 4z = 0\} , & S_2 &= \{(x, y, z) \mid 2x + 3y + 4z = 1\} , \\ S_3 &= \{(x, y, z) \mid 2x + 3y + 4z \leq 0\} , & S_4 &= \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\} . \end{aligned}$$

Describe each of these sets geometrically and decide whether it is a subspace of $\mathbb{R}^3$.

4. Working over $\mathbb{R}$, determine whether the following matrices have the same or different row spaces:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{bmatrix} , \quad B = \begin{bmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{bmatrix} , \quad C = \begin{bmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{bmatrix} ,$$

5. Let $S_1 = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_2 \rangle$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = (1, 2, -1, 3) , \quad \mathbf{v}_2 = (2, 4, 1, -2) , \quad \mathbf{v}_3 = (3, 6, 3, -7) ,$$

and $S_2 = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{w}_1 = (1, 2, -4, 11) , \quad \mathbf{w}_2 = (2, 4, -5, 14) .$$

By row reducing appropriate matrices, verify that $S_1 = S_2$.

6.* Let V be a vector space over a field F. Prove carefully from the definition of a vector space the following elementary properties:

(a) The zero vector is unique.

(b) The negative of a vector is unique.

(c) For all $\mathbf{v} \in V$, we have $0\mathbf{v} = \mathbf{0}$, where 0 is the zero in F and $\mathbf{0}$ is the zero vector.

(d) For all $\lambda \in F$, we have $\lambda\mathbf{0} = \mathbf{0}$.

(e) For all $\mathbf{v} \in V$, we have $(-1)\mathbf{v} = -\mathbf{v}$, the negative vector.

(f) For all $\mathbf{v} \in V$ and $\lambda \in F$, we have that $\lambda\mathbf{v} = \mathbf{0}$ implies $\lambda = 0$ or $\mathbf{v} = \mathbf{0}$.

Further Exercises:

7. Let V be a vector space over a field F. Verify that S is closed under addition and scalar multiplication if and only if $\lambda\mathbf{v} + \mu\mathbf{w} \in S$ for all $\mathbf{v}, \mathbf{w} \in S$ and $\lambda, \mu \in F$ (that is, S is closed under taking linear combinations).

8. Let V be a vector space and suppose that S and T are subspaces of V. Verify that the intersection $S \cap T$ is a subspace of V.

9. Explain why a subspace of a vector space is a vector space in its own right, that is, becomes an abelian group with a compatible scalar multiplication.

10. Identify the zero vector and negative vectors in the vector space F^X of functions from X to F, where F is any field and X any nonempty set.

11. Let $m, n \geq 1$ and M be an $m \times n$ matrix over a field F. Verify that the null space of M, namely

$$S \;=\; \{\mathbf{v} \in F^n \mid M\mathbf{v}^\top = \mathbf{0}\} \;,$$

is a subspace of F^n.

12.* Let V be any vector space. Verify that every subspace of V contains the zero vector $\mathbf{0}$ and that $\{\mathbf{0}\}$ is a subspace of V. Deduce that

$$\{\mathbf{0}\} \;=\; \bigcap \{S \mid S \text{ is a subspace of } V \text{ containing the empty set}\} \;.$$

This explains why we define the span of the empty set to be the trivial subspace (and explains why, after we introduce the concepts of *basis* and *dimension*, that the trivial vector space is *zero-dimensional*).

13.* A square matrix M is *symmetric* if it equals its transpose, that is, $M = M^\top$. Verify that, for $n \geq 1$, and working over some field F, the set S of symmetric $n \times n$ matrices forms a subspace of the vector space $\mathrm{Mat}_{n,n}(F)$ of $n \times n$ matrices over F. Find a spanning set for S if $n = 2$.

14.* Consider the field $F = \mathbb{R}$. Recall that $\mathbb{P}_n$ denotes the vector space of all real polynomials (which may also be regarded as real polynomial functions) of degree at most n, where $n \geq 0$. Now put

$$\mathbb{P} \;=\; \bigcup_{n \geq 0} \mathbb{P}_n \;.$$

(a) Verify that $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ for each n, and that $\mathbb{P}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

(b) Explain why $\{1, x, x^2, \ldots, x^n\}$ spans $\mathbb{P}_n$.

(c) Explain why $\mathbb{P}_n$ and $\mathbb{R}^{n+1}$ are isomorphic as vector spaces for each $n \geq 0$.

(d) Explain why no finite subset of $\mathbb{P}$ can span $\mathbb{P}$.

15.* A function $f : \mathbb{R} \to \mathbb{R}$ is called *bounded* if there exists some nonnegative real number K such that $|f(x)| \leq K$ for all $x \in \mathbb{R}$. Prove that the set of all bounded functions is a subspace of the vector space $\mathbb{R}^{\mathbb{R}}$ of all real valued real functions.