The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 | **Exercises for Week 12 (beginning 20 May)** | 2019

**Important Ideas and Useful Facts:**

(i) The Gram-Schmidt orthogonalisation process: It is an important theorem that every inner product space $V$ possesses an orthonormal basis. The proof involves a recursive construction known as the *Gram-Schmidt process* which proceeds as follows. Choose any basis $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ for $V$, which certainly exists, and now take the following steps:

*Step* (1)*:* Put $\mathbf{v}_1 = \dfrac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$, which is a normal vector spanning $\langle \mathbf{u}_1 \rangle$.

*Step* (2)*:* Put $W = \langle \mathbf{v}_1 \rangle = \langle \mathbf{u}_1 \rangle$ and

$$\mathbf{v}_2 \;=\; \frac{\mathbf{u}_2 - \mathrm{Proj}_W \mathbf{u}_2}{\|\mathbf{u}_2 - \mathrm{Proj}_W \mathbf{u}_2\|} \;=\; \frac{\mathbf{u}_2 - \langle \mathbf{u}_2, \mathbf{v}_1 \rangle \mathbf{v}_1}{\|\mathbf{u}_2 - \langle \mathbf{u}_2, \mathbf{v}_1 \rangle \mathbf{v}_1\|} \;,$$

which is a normal vector orthogonal to $\mathbf{v}_1$. Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for $\langle \mathbf{v}_1, \mathbf{v_2} \rangle = \langle \mathbf{u}_1, \mathbf{u_2} \rangle$.

*Step* (3)*:* Put $W = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ and

$$\mathbf{v}_3 \;=\; \frac{\mathbf{u}_3 - \mathrm{Proj}_W \mathbf{u}_3}{\|\mathbf{u}_3 - \mathrm{Proj}_W \mathbf{u}_3\|} \;=\; \frac{\mathbf{u}_3 - \langle \mathbf{u}_3, \mathbf{v}_1 \rangle \mathbf{v}_1 - \langle \mathbf{u}_3, \mathbf{v}_2 \rangle \mathbf{v}_2}{\|\mathbf{u}_3 - \langle \mathbf{u}_3, \mathbf{v}_1 \rangle \mathbf{v}_1 - \langle \mathbf{u}_3, \mathbf{v}_2 \rangle \mathbf{v}_2\|} \;,$$

which is a normal vector orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle$.

*Step* ($i$+1)*:* Suppose that $i < n$ and we have constructed an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ for $W = \langle \mathbf{v}_1, \ldots, \mathbf{v}_i \rangle = \langle \mathbf{u}_1, \ldots, \mathbf{u}_i \rangle$. Put

$$\mathbf{v}_{i+1} \;=\; \frac{\mathbf{u}_{i+1} - \mathrm{Proj}_W \mathbf{u}_{i+1}}{\|\mathbf{u}_{i+1} - \mathrm{Proj}_W \mathbf{u}_{i+1}\|} \;=\; \frac{\mathbf{u}_{i+1} - \langle \mathbf{u}_{i+1}, \mathbf{v}_1 \rangle \mathbf{v}_1 - \ldots - \langle \mathbf{u}_{i+1}, \mathbf{v}_i \rangle \mathbf{v}_i}{\|\mathbf{u}_{i+1} - \langle \mathbf{u}_{i+1}, \mathbf{v}_1 \rangle \mathbf{v}_1 - \ldots - \langle \mathbf{u}_{i+1}, \mathbf{v}_i \rangle \mathbf{v}_i\|} \;,$$

which is a normal vector orthogonal to $\mathbf{v}_1, \ldots, \mathbf{v}_i$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_i, \mathbf{v}_{i+1}\}$ is an orthonormal basis for $\langle \mathbf{v}_1, \ldots \mathbf{v}_i, \mathbf{v}_{i+1} \rangle = \langle \mathbf{u}_1, \ldots, \mathbf{u}_i, \mathbf{u}_{i+1} \rangle$.

Continue these steps until $i + 1 = n$, and then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ will be an orthonormal basis for $V$.

(ii) The $QR$ factorisation of a matrix: An application of the Gram-Schmidt process is that every $m \times n$ matrix $M$ with linearly independent columns (so necessarily $m \geq n$) has a factorisation

$$M \;=\; QR$$

where the columns of $Q$ form an orthonormal basis for the column space of $M$ and $R$ is an invertible upper triangular matrix. The diagonal entries of $R$ can be chosen to be positive.

(iii) The Spectral Theorem for real symmetric matrices: Let $M$ be an $n \times n$ real matrix. If $M$ is *symmetric*, that is $M = M^T$, then the eigenvalues of $M$ are all real. The Spectral Theorem states that $M$ is symmetric if and only if $M$ is *orthogonally diagonalisable*, that is, there exists an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $Q^T M Q = D$.

(iv) Adjoint of a real linear operator: Let $L : V \to V$ be a linear operator where $V$ is an inner product space. Then there exists a unique operator $L^* : V \to V$, called the *adjoint* of $L$, such that

$$\langle L(\mathbf{u}), \mathbf{v} \rangle \;=\; \langle \mathbf{u}, L^*(\mathbf{v}) \rangle$$

for all $\mathbf{u}, \mathbf{v} \in V$. If $B$ is an orthonormal basis for $V$ then the matrix of $L^*$ with respect to $B$ is the transpose of the matrix of $L$ with respect to $B$, that is,

$$[L^*]_B^B \;=\; \left([L]_B^B\right)^T .$$

We call the operator $L$ *symmetric* if $L = L^*$, which is equivalent to the matrix of $L$ with respect to any orthonormal basis for $V$ being symmetric.

(v) Linear combinations of operators: If $L_1, \ldots, L_k : V \to V$ are linear operators of a vector space $V$ over a field $F$ and $\lambda_1, \ldots, \lambda_k \in F$, then the *linear combination* $\lambda_1 L_1 + \ldots + \lambda_k L_k : V \to V$ is the linear operator defined by the following rule, for all $\mathbf{v} \in V$:

$$(\lambda_1 L_1 + \ldots + \lambda_k L_k)(\mathbf{v}) \;=\; \lambda_1 L_1(\mathbf{v}) + \ldots + \lambda_k L_k(\mathbf{v}) .$$

(vi) The Spectral Theorem for real symmetric linear operators: Let $L : V \to V$ be a symmetric linear operator where $V$ is an inner product space. Then there exists a direct sum decomposition

$$V \;=\; W_1 \oplus \ldots \oplus W_k$$

where $W_1, \ldots, W_k$ are subspaces of $V$ which are pairwise mutually orthogonal (that is $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ whenever $\mathbf{v} \in W_i$ and $\mathbf{w} \in W_j$ for $i \neq j$), and scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ (in fact the eigenvalues of $L$) such that $L$ is the following linear combination of projection operators:

$$L \;=\; \lambda_1 \mathrm{Proj}_{W_1} + \ldots + \lambda_k \mathrm{Proj}_{W_k} .$$

Here the projection operators are mutually orthogonal, in the sense that the composite of any distinct pair of projection operators is the *zero operator* (that maps everything to zero), and they sum (as a linear combination) to the identity operator.

**Tutorial Exercises:**

**1.** Let $W$ be the plane defined by the equation $x + 2y - z = 0$.

(a) Verify that $\mathbf{v}_1 = (1, 0, 1)$ and $\mathbf{v}_2 = (-2, 1, 0)$ span $W$, that is,

$$W \;=\; \{\lambda_1\mathbf{v}_1 + \lambda_1\mathbf{v}_2 \mid \lambda_1, \lambda_2 \in \mathbb{R}\}\ .$$

(b) Apply the Gram-Schmidt process (in just two stages) to produce an orthonormal basis for $W$.

**2.** Consider the subspace $W = \{(x, y, z, w) \mid x - y + z - w = 0\}$ of $\mathbb{R}^4$. Let $\mathbf{v} = (1, 2, 3, 4)$.

(a) Verify that $\mathbf{v}_1 = (1, 1, 0, 0)$, $\mathbf{v}_2 = (-1, 0, 1, 0)$ and $\mathbf{v}_3 = (1, 0, 0, 1)$ span $W$.

(b) Apply the Gram-Schmidt process to obtain an orthonormal basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ for $W$.

(c) Find $\text{proj}_W\mathbf{v} \;=\; \langle\mathbf{v}, \mathbf{b}_1\rangle\mathbf{b}_1 + \langle\mathbf{v}, \mathbf{b}_2\rangle\mathbf{b}_2 + \langle\mathbf{v}, \mathbf{b}_3\rangle\mathbf{b}_3$ , where the inner product is the usual dot product.

(d) Find the distance from $\mathbf{v}$ to $W$ and the closest point on $W$ to $\mathbf{v}$.

**3.** Let $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be an orthonormal subset of an inner product space. State the Generalised Theorem of Pythagoras, and use it to find each of the following:

(a) $\|3\mathbf{b}_1 + 4\mathbf{b}_2\|$ (b) $\|\mathbf{b}_1 + 2\mathbf{b}_2 - 2\mathbf{b}_3\|$ (c) $\|4\mathbf{b}_1 - \mathbf{b}_2 + 8\mathbf{b}_3\|$

**4.*** Recall that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are called *linearly independent* if, for all scalars $\lambda_1, \ldots, \lambda_n$,

$$\lambda_1\mathbf{v}_1 + \ldots + \lambda_n\mathbf{v}_n \;=\; \mathbf{0} \quad\Longrightarrow\quad \lambda_1 = \ldots = \lambda_n = 0\ .$$

Verify that any nonzero pairwise orthogonal vectors are linearly independent.

**5.*** Let $V$ be the inner product space of continuous functions: $[-\pi, \pi] \to \mathbb{R}$ with inner product

$$\langle f, g\rangle \;=\; \int_{-\pi}^{\pi} f(x)g(x)\,dx\ .$$

(a) Find the distance from $\cos x$ to $\sin x$ in $V$.

(b) Find $\|\cos nx\|$ and $\|\sin mx\|$ when $n \geq 0$ and $m \geq 1$.

(c) Verify that $\cos nx$ is orthogonal to $\cos mx$ if $m, n \geq 0$, $m \neq n$.

(d) Verify that $\sin nx$ is orthogonal to $\sin mx$ if $m, n \geq 1$, $m \neq n$.

(e) Verify that $\cos nx$ is orthogonal to $\sin mx$ for all $n \geq 0$, $m \geq 1$.

(f) Use the Generalised Theorem of Pythagoras to deduce quickly the distance from $\cos nx$ to $\sin nx$ for any $n \geq 1$.

**Further Exercises:**

**6.** Let $\mathbf{v} = (0, 2, 1, -3)$ and let

$$W = \{(x, y, z, w) \mid x + y + 2z - w = 0\}$$

which is spanned by $\mathbf{v}_1 = (-1, 1, 0, 0)$, $\mathbf{v}_2 = (-2, 0, 1, 0)$, $\mathbf{v}_3 = (1, 0, 0, 1)$.

(a) Apply the Gram-Schmidt process to obtain an orthonormal basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ for $W$.

(b) Find $\text{proj}_W \mathbf{v} = \langle \mathbf{v}, \mathbf{b}_1 \rangle \mathbf{b}_1 + \langle \mathbf{v}, \mathbf{b}_2 \rangle \mathbf{b}_2 + \langle \mathbf{v}, \mathbf{b}_3 \rangle \mathbf{b}_3$ .

(c) Find the distance from $\mathbf{v}$ to $W$ and the closest point on $W$ to $\mathbf{v}$.

**7.*** Let $V$ be the inner product space of continuous functions: $[-1, 1] \to \mathbb{R}$ with inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx \,.$$

Let $W = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_1, a_1, a_2, a_3 \in \mathbb{R}\}$ be the subspace of $V$ spanned by $1, x, x^2, x^3$. Find an orthonormal basis for $W$ (obtaining the first four *normalised Legendre polynomials*).

**8.*** Let $V$ be the space of integrable periodic functions with period $2\pi$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx$. Let $k$ be a positive integer. By the last tutorial exercise, we have that

$$B = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \ldots, \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \ldots, \frac{\sin kx}{\sqrt{\pi}} \right\}$$

is an orthonormal set. It is a basis for the subspace $W_k$ of trigonometric polynomials of degree $\leq k$. Thus if $f \in V$ then

$$\text{proj}_{W_k} f = a_0 + \sum_{n=1}^{k} a_n \cos nx + b_n \sin nx$$

where, for $1 \leq n \leq k$,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \,, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \,, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \,.$$

(a) Find $\text{proj}_{W_k} f$ where $f$ is the periodic saw tooth function satisfying $f(x) = x$ for $x \in (-\pi, \pi]$.

(b) Let $k \to \infty$ and, by taking $x = \pi/2$, deduce the famous series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \,.$$

**9.*** As in the previous question, let $V$ be the space of integrable periodic functions with period $2\pi$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx$. The *Fourier series* of $f \in V$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where $a_0$, $a_n$ and $b_n$, for $n \geq 1$, are as defined in the previous exercise. I

(a) Let $f \in V$ be even and $g \in V$ be odd. Explain briefly why $f$ and $g$ are orthogonal.

(b) Find the Fourier series for the piecewise continuous even periodic function $f$ satisfying

$$f(x) = x^2 \qquad \text{for } -\pi \leq x \leq \pi.$$

(c) Deduce from your answer to (b) the famous series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \,.$$

You may assume convergence of the Fourier series to the function.