The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1	**Week 12 Longer Solutions**	2019

1. (a) We have

$$\begin{aligned} W &= \{(x,y,z) \mid x+2y-z=0\} \;=\; \{(-2y+z,y,z) \mid y,z\in\mathbb{R}\} \\ &= \{y(-2,1,0)+z(1,0,1) \mid y,z\in\mathbb{R}\} \;=\; \{\lambda_1(1,0,1)+\lambda_2(-2,1,0) \mid \lambda_1,\lambda_2\in\mathbb{R}\} \\ &= \{\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2 \mid \lambda_1,\lambda_2\in\mathbb{R}\} = \langle \mathbf{v}_1,\mathbf{v}_2\rangle \;, \end{aligned}$$

which verifies that $\mathbf{v}_1$ and $\mathbf{v}_2$ span W.

(b) First stage: observe that $\|\mathbf{v}_1\| = \sqrt{1+0+1} = \sqrt{2}$, so that

$$\mathbf{b}_1 \;=\; \widehat{\mathbf{v}}_1 \;=\; \frac{1}{\sqrt{2}}(1,0,1)\;.$$

Second stage:

$$\begin{aligned} \mathbf{w}_2 &\;=\; \mathbf{v}_2-(\mathbf{v}_2\cdot\mathbf{b}_1)\mathbf{b}_1 \;=\; (-2,1,0)-\frac{-2}{2}(1,0,1) \;=\; (-1,1,1)\;, \\ \mathbf{b}_2 &\;=\; \widehat{\mathbf{w}}_2 \;=\; \frac{1}{\sqrt{3}}(-1,1,1)\;. \end{aligned}$$

Thus an orthonormal basis for W is $\left\{\frac{1}{\sqrt{2}}(1,0,1)\,,\;\frac{1}{\sqrt{3}}(-1,1,1)\right\}$.

2. (a) We have

$$\begin{aligned} W &= \{(x,y,z,w) \mid x-y+z-w=0\} \;=\; \{(y-z+w,y,z,w) \mid y,z,w\in\mathbb{R}\} \\ &= \{y(1,1,0,0)+z(-1,0,1,0)+w(1,0,0,1) \mid y,z,w\in\mathbb{R}\} \\ &= \{\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2+\lambda_3\mathbf{v}_3 \mid \lambda_1,\lambda_2,\lambda_3\in\mathbb{R}\} = \langle \mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\rangle \;, \end{aligned}$$

which verifies that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ span W.

(b) First stage: observe that $\|\mathbf{v}_1\| = \sqrt{1+1+0+0} = \sqrt{2}$, so that

$$\mathbf{b}_1 \;=\; \widehat{\mathbf{v}}_1 \;=\; \frac{1}{\sqrt{2}}(1,1,0,0)\;.$$

Second stage:

$$\begin{aligned} \mathbf{w}_2 &\;=\; \mathbf{v}_2-(\mathbf{v}_2\cdot\mathbf{b}_1)\mathbf{b}_1 \;=\; (-1,0,1,0)-\frac{-1}{2}(1,1,0,0) \;=\; \frac{1}{2}(-1,1,2,0)\;, \\ \mathbf{b}_2 &\;=\; \widehat{\mathbf{w}}_2 \;=\; \frac{1}{\sqrt{6}}(-1,1,2,0)\;. \end{aligned}$$

Third stage:

$$\begin{aligned}\mathbf{w}_3 &= \mathbf{v}_3 - (\mathbf{v}_3\cdot\mathbf{b}_1)\mathbf{b}_1 - (\mathbf{v}_3\cdot\mathbf{b}_2)\mathbf{b}_2\\ &= (1,0,0,1) - \frac{1}{2}(1,1,0,0) - \frac{-1}{6}(-1,1,2,0) \;=\; \frac{1}{3}(1,-1,1,3)\ ,\\ \mathbf{b}_3 &= \widehat{\mathbf{w}}_3 \;=\; \frac{1}{2\sqrt{3}}(1,-1,1,3)\ .\end{aligned}$$

Thus an orthonormal basis for W is

$$\left\{\frac{1}{\sqrt{2}}(1,1,0,0)\ ,\ \frac{1}{\sqrt{6}}(-1,1,2,0)\ ,\ \frac{1}{2\sqrt{3}}(1,-1,1,3)\right\}\ .$$

(c) We have

$$\begin{aligned}\mathrm{proj}_W\mathbf{v} &= (\mathbf{v}\cdot\mathbf{b}_1)\mathbf{b}_1 + (\mathbf{v}\cdot\mathbf{b}_2)\mathbf{b}_2 + (\mathbf{v}\cdot\mathbf{b}_3)\mathbf{b}_3\\ &= \frac{3}{2}(1,1,0,0) + \frac{7}{6}(-1,1,2,0) + \frac{14}{12}(1,-1,1,3)\\ &= \left(\frac{3}{2},\frac{3}{2},\frac{7}{2},\frac{7}{2}\right)\ .\end{aligned}$$

(d) Thus the closest point on W to $\mathbf{v}$ is $(\frac{3}{2},\frac{3}{2},\frac{7}{2},\frac{7}{2})$, and the shortest distance is

$$\|(1,2,3,4) - (\tfrac{3}{2},\tfrac{3}{2},\tfrac{7}{2},\tfrac{7}{2})\| \;=\; \|(-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})\| \;=\; 1\ .$$

3. The Generalised Theorem of Pythagoras states that if $\mathbf{v}_1,\ldots,\mathbf{v}_n$ are pairwise orthogonal vectors in an inner product space, where $n\geq 2$, then

$$\|\mathbf{v}_1+\ldots+\mathbf{v}_n\|^2 \;=\; \|\mathbf{v}_1\|^2+\ldots+\|\mathbf{v}_n\|^2\ .$$

(a) We have

$$\|3\mathbf{b}_1+4\mathbf{b}_2\| \;=\; \sqrt{\|3\mathbf{b}_1\|^2+\|4\mathbf{b}_2\|^2} \;=\; \sqrt{9\|\mathbf{b}_1\|^2+16\|\mathbf{b}_2\|^2} \;=\; \sqrt{25} \;=\; 5\ .$$

(b) We have

$$\begin{aligned}\|\mathbf{b}_1+2\mathbf{b}_2-2\mathbf{b}_3\| &= \sqrt{\|\mathbf{b}_1\|^2+\|2\mathbf{b}_2\|^2+\|-2\mathbf{b}_3\|^2}\\ &= \sqrt{1+4\|\mathbf{b}_2\|^2+4\|\mathbf{b}_3\|^2} \;=\; \sqrt{1+4+4} \;=\; \sqrt{9} \;=\; 3\ .\end{aligned}$$

(c) We have

$$\begin{aligned}\|4\mathbf{b}_1-\mathbf{b}_2+8\mathbf{b}_3\| &= \sqrt{\|4\mathbf{b}_1\|^2+\|-\mathbf{b}_2\|^2+\|8\mathbf{b}_3\|^2} \;=\; \sqrt{16\|\mathbf{b}_1\|^2+\|\mathbf{b}_2\|^2+64\|\mathbf{b}_3\|^2}\\ &= \sqrt{16+1+64} \;=\; \sqrt{81} \;=\; 9\ .\end{aligned}$$

4. Let $\mathbf{v}_1,\ldots,\mathbf{v}_n\in X$ be pairwise orthogonal nonzero vectors. Suppose $\lambda_1,\ldots,\lambda_n\in\mathbb{R}$ such that $\lambda_1\mathbf{v}_1+\ldots+\lambda_n\mathbf{v}_n=\mathbf{0}$. Then, for each $i=1,\ldots,n$, we have, by elementary properties of the inner product and by orthogonality,

$$\begin{aligned}0 &= \langle\mathbf{0},\mathbf{v}_i\rangle \;=\; \langle\lambda_1\mathbf{v}_1+\ldots+\lambda_n\mathbf{v}_n,\mathbf{v}_i\rangle\\ &= \lambda_1\langle\mathbf{v}_1,\mathbf{v}_i\rangle+\ldots+\lambda_{i-1}\langle\mathbf{v}_i,\mathbf{v}_i\rangle+\lambda_i\langle\mathbf{v}_i,\mathbf{v}_i\rangle+\lambda_{i+1}\langle\mathbf{v}_{i+1},\mathbf{v}_i\rangle+\ldots+\lambda_n\langle\mathbf{v}_n,\mathbf{v}_i\rangle\\ &= \lambda_1(0)+\ldots+\lambda_{i-1}(0)+\lambda_i\|\mathbf{v}_i\|^2+\lambda_{i+1}(0)+\ldots+\lambda_n(0)\\ &= \lambda_i\|\mathbf{v}_i\|^2\ ,\end{aligned}$$

so that $\lambda_i=0$, since $\|\mathbf{v}_i\|\neq 0$, as $\mathbf{v}_i$ is nonzero. This shows that $\lambda_1=\ldots=\lambda_n=0$, and completes the proof that X is linearly independent.

5. (a) We have

$$\begin{aligned}\|\cos x - \sin x\|^2 &= \int_{-\pi}^{\pi} (\cos x - \sin x)^2\,dx = \int_{-\pi}^{\pi} \cos^2 x - 2\cos x \sin x + \sin^2 x\,dx \\ &= \int_{-\pi}^{\pi} 1 - \sin 2x\,dx = \left[x + \frac{\cos 2x}{2}\right]_{-\pi}^{\pi} = 2\pi\ ,\end{aligned}$$

so that the distance from $\cos x$ to $\sin x$ in V is $\sqrt{2\pi}$.

(b) We have $\|\cos 0x\|^2 = \int_{-\pi}^{\pi} 1\,dx = 2\pi$, so $\|\cos 0x\| = \sqrt{2\pi}$. For $n \geq 1$,

$$\|\cos nx\|^2 = \int_{-\pi}^{\pi} \cos^2 nx\,dx = \int_{-\pi}^{\pi} \frac{1 + \cos 2nx}{2}\,dx = \left[\frac{x}{2} + \frac{\sin 2nx}{4n}\right]_{-\pi}^{\pi} = \pi$$

and

$$\|\sin nx\|^2 = \int_{-\pi}^{\pi} \sin^2 nx\,dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2nx}{2}\,dx = \left[\frac{x}{2} - \frac{\sin 2nx}{4n}\right]_{-\pi}^{\pi} = \pi\ ,$$

so that $\|\cos nx\| = \|\sin nx\| = \sqrt{\pi}$.

(c) Let $m, n \geq 1, m \neq n$. Then

$$\langle \cos 0x, \cos nx\rangle = \int_{-\pi}^{\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} = 0\ ,$$

and

$$\begin{aligned}\langle \cos mx, \cos nx\rangle &= \int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x + \cos(m-n)x\,dx \\ &= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0\ .\end{aligned}$$

(d) Let $m, n \geq 1, m \neq n$. Then

$$\begin{aligned}\langle \sin mx, \sin nx\rangle &= \int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)x - \cos(m+n)x\,dx \\ &= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0\ .\end{aligned}$$

(e) If $n \geq 0$ and $m \geq 1$ then

$$\langle \cos nx, \sin mx\rangle = \int_{-\pi}^{\pi} \cos nx \sin mx\,dx = 0\ ,$$

since the integrand is an odd function.

(f) By the Generalised Theorem of Pythagoras, for $n \geq 1$, since the functions are orthogonal,

$$\|\cos nx - \sin nx\|^2 = \|\cos nx\|^2 + \|\sin nx\|^2 = \pi + \pi = 2\pi\ ,$$

so the distance is $\sqrt{2\pi}$.

6. (a) First stage:

$$\mathbf{b}_1 \;=\; \widehat{\mathbf{v}}_1 \;=\; \frac{1}{\sqrt{2}}(-1,1,0,0)\ .$$

Second stage:

$$\begin{aligned}
\mathbf{w}_2 \;&=\; \mathbf{v}_2 - (\mathbf{v}_2\cdot\mathbf{b}_1)\mathbf{b}_1 \;=\; (-2,0,1,0) - \frac{2}{2}(-1,1,0,0) \;=\; (-1,-1,1,0)\ ,\\
\mathbf{b}_2 \;&=\; \widehat{\mathbf{w}}_2 \;=\; \frac{1}{\sqrt{3}}(-1,-1,1,0)\ .
\end{aligned}$$

Third stage:

$$\begin{aligned}
\mathbf{w}_3 \;&=\; \mathbf{v}_3 - (\mathbf{v}_3\cdot\mathbf{b}_1)\mathbf{b}_1 - (\mathbf{v}_3\cdot\mathbf{b}_2)\mathbf{b}_2\\
&=\; (1,0,0,1) - \frac{-1}{2}(-1,1,0,0) - \frac{-1}{3}(-1,-1,1,0) \;=\; \frac{1}{6}(1,1,2,6)\ ,\\
\mathbf{b}_3 \;&=\; \widehat{\mathbf{w}}_3 \;=\; \frac{1}{\sqrt{42}}(1,1,2,6)\ .
\end{aligned}$$

Thus an orthonormal basis for W is

$$\left\{\frac{1}{\sqrt{2}}(-1,1,0,0)\,,\ \frac{1}{\sqrt{3}}(-1,-1,1,0)\,,\ \frac{1}{\sqrt{42}}(1,1,2,6)\right\}.$$

(b) We have

$$\begin{aligned}
\mathrm{proj}_W\mathbf{v} \;&=\; (\mathbf{v}\cdot\mathbf{b}_1)\mathbf{b}_1 + (\mathbf{v}\cdot\mathbf{b}_2)\mathbf{b}_2 + (\mathbf{v}\cdot\mathbf{b}_3)\mathbf{b}_3\\
&=\; \frac{2}{2}(-1,1,0,0) + \frac{-1}{3}(-1,-1,1,0) + \frac{-14}{42}(1,1,2,6)\\
&=\; (-1,1,-1,-2)\ .
\end{aligned}$$

(c) Thus the closest point on W to $\mathbf{v}$ is $(-1,1,-1,-2)$, and the shortest distance is

$$\|(0,2,1,-3)-(-1,1,-1,-2)\| \;=\; \|(1,1,2,-1)\| \;=\; \sqrt{7}\ .$$

7. Put $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$, $\mathbf{v}_3 = x^2$ and $\mathbf{v}_4 = x^3$. Notice that $\{\mathbf{v}_1,\mathbf{v}_2\}$, $\{\mathbf{v}_1,\mathbf{v}_4\}$, $\{\mathbf{v}_2,\mathbf{v}_3\}$ and $\{\mathbf{v}_3,\mathbf{v}_4\}$ are orthogonal pairs, since the sum of the exponents as powers of x is odd in each case. This simplifies the following calculations.

First stage: observe that $\|1\| = \sqrt{\int_{-1}^{1} 1^2\,dx} = \sqrt{2}$, so that

$$\mathbf{b}_1 \;=\; \widehat{\mathbf{v}}_1 \;=\; \frac{1}{\sqrt{2}}\ .$$

Second stage:

$$\begin{aligned}
\mathbf{w}_2 \;&=\; \mathbf{v}_2 - \langle\mathbf{v}_2,\mathbf{b}_1\rangle\mathbf{b}_1 \;=\; \mathbf{v}_2\ ,\\
\mathbf{b}_2 \;&=\; \widehat{\mathbf{w}}_2 \;=\; \frac{x}{\|x\|} \;=\; \frac{x}{\sqrt{\int_{-1}^{1} x^2\,dx}} \;=\; \frac{x}{\sqrt{2/3}} \;=\; \sqrt{\frac{3}{2}}\,x\ .
\end{aligned}$$

Third stage:

$$\begin{aligned}
\mathbf{w}_3 &= \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{b}_1 \rangle \mathbf{b}_1 - \langle \mathbf{v}_3, \mathbf{b}_2 \rangle \mathbf{b}_2 \;=\; \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{b}_1 \rangle \mathbf{b}_1 \\
&= x^2 - \left(\int_{-1}^{1} \frac{x^2}{\sqrt{2}}\, dx \right) \frac{1}{\sqrt{2}} \;=\; x^2 - \frac{1}{2} \int_{-1}^{1} x^2\, dx \;=\; x^2 - \frac{1}{3}\,, \\
\mathbf{b}_3 &= \widehat{\mathbf{w}}_3 \;=\; \frac{x^2 - 1/3}{\|x^2 - 1/3\|} \;=\; \frac{x^2 - 1/3}{\sqrt{\int_{-1}^{1} (x^2 - 1/3)^2\, dx}} \\
&= \frac{x^2 - 1/3}{\sqrt{\int_{-1}^{1} x^4 - 2x^2/3 + 1/9\, dx}} \;=\; \frac{x^2 - 1/3}{\sqrt{8/45}} \;=\; \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)\,.
\end{aligned}$$

Fourth stage:

$$\begin{aligned}
\mathbf{w}_4 &= \mathbf{v}_4 - \langle \mathbf{v}_4, \mathbf{b}_1 \rangle \mathbf{b}_1 - \langle \mathbf{v}_4, \mathbf{b}_2 \rangle \mathbf{b}_2 - \langle \mathbf{v}_4, \mathbf{b}_3 \rangle \mathbf{b}_3 \;=\; \mathbf{v}_4 - \langle \mathbf{v}_4, \mathbf{b}_2 \rangle \mathbf{b}_2 \\
&= x^3 - \sqrt{\frac{3}{2}} \left(\int_{-1}^{1} x^4\, dx \right) \sqrt{\frac{3}{2}}\, x \;=\; x^3 - \frac{3x}{2} \int_{-1}^{1} x^4\, dx \;=\; x^3 - \frac{3x}{5}\,, \\
\mathbf{b}_4 &= \widehat{\mathbf{w}}_4 \;=\; \frac{x^3 - 3x/5}{\|x^3 - 3x/5\|} \;=\; \frac{x^3 - 3x/5}{\sqrt{\int_{-1}^{1} (x^3 - 3x/5)^2\, dx}} \\
&= \frac{x^3 - 3x/5}{\sqrt{\int_{-1}^{1} x^6 - 6x^4/5 + 9x^2/25\, dx}} \;=\; \frac{x^3 - 3x/5}{\sqrt{8/175}} \;=\; \frac{\sqrt{7}}{2\sqrt{2}}(5x^3 - 3x)\,.
\end{aligned}$$

Thus an orthonormal basis for W is

$$\left\{ \frac{1}{\sqrt{2}}\,,\; \frac{\sqrt{3}\,x}{\sqrt{2}}\,,\; \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)\,,\; \frac{\sqrt{7}}{2\sqrt{2}}(5x^3 - 3x) \right\}.$$

8. (a) Observe first that $y = f(x)\cos mx$ is an odd function for $x \in (-\pi, \pi)$ and $m \geq 0$, so that

$$a_0 \;=\; a_1 \;=\; \ldots \;=\; a_k \;=\; 0\,.$$

Observe next that $y = f(x)\sin nx$ is an even function for $x \in (-\pi, \pi)$ and $n \geq 1$, so that

$$\begin{aligned}
b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, dx \;=\; \frac{2}{\pi} \int_0^{\pi} x \sin nx\, dx \\
&= \frac{2}{\pi} \left[-\frac{x \cos nx}{n} \right]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} \frac{-\cos nx}{n}\, dx \\
&= -\frac{2\cos n\pi}{n} + \frac{2}{n^2 \pi} \big[\sin nx \big]_0^{\pi} \\
&= \frac{2}{n}(-1)^{n-1}\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\operatorname{proj}_{W_k} f &= 2 \sum_{n=1}^{k} \frac{(-1)^{n-1}}{n} \sin nx \\
&= 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots + \frac{(-1)^{k-1} \sin kx}{k} \right).
\end{aligned}$$

(b) Letting $k \to \infty$ gives the Fourier series

$$f(x) \;=\; 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right).$$

Putting $x = \pi/2$ gives

$$\frac{\pi}{2} \;=\; 2\left(1 - \frac{0}{2} + \frac{-1}{3} - \frac{0}{4} + \frac{1}{5} - \frac{0}{6} + \frac{-1}{7} + \ldots\right),$$

so that, finally,

$$\frac{\pi}{4} \;=\; 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\,.$$

9. (a) Let h be defined by $h(x) = f(x)g(x)$. Then h is odd, being a product of an odd and an even function, so that

$$\langle f, g\rangle \;=\; \int_{-\pi}^{\pi} f(x)g(x)\,dx \;=\; \int_{-\pi}^{\pi} h(x)\,dx \;=\; 0\,,$$

so that f and g are orthogonal.

(b) Observe that $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx \;=\; \pi^2/3$, and, for $n \geq 1$,

$$a_n \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx \;=\; \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_{-\pi}^{\pi} \;=\; \frac{4}{n^2}(-1)^n\,,$$

and $b_n = 0$, since f is even and the sine function is odd. Hence

$$f(x) \;=\; \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx\,.$$

(c) From part (b), in particular,

$$\pi^2 \;=\; f(\pi) \;=\; \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos n\pi \;=\; \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\,,$$

giving $\dfrac{2\pi^2}{3} \;=\; \displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2}$, so that, finally,

$$\frac{\pi^2}{6} \;=\; 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\,.$$