The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Revision Exercises for Week 13 (beginning 27 May)** 2019

1. Consider the following real matrix:

$$M = \begin{bmatrix} -9 & 8 & 0 \\ -12 & 11 & 0 \\ -8 & 6 & 1 \end{bmatrix}$$

(a) Working over $\mathbb{R}$, find the characteristic polynomial of M.

(b) How do you know immediately, as a result of your calculation in (a), that M is invertible.

(c) Use the result of part (a), and the Cayley-Hamilton Theorem, to find M^{-1} as a quadratic expression in M. Now use this expression to evaluate M^{-1}.

(d) Use part (c) or otherwise to evaluate M^{-1} when working instead over the field $\mathbb{Z}_{13}$.

(e) Working over $\mathbb{R}$, find the eigenvalues of M and the corresponding eigenspaces.

(f) Working over $\mathbb{R}$, find an invertible matrix P and a diagonal matrix D such that

$$M = PDP^{-1} .$$

(g) Consider three differentiable functions

$$x = x(t) , \quad y = y(t) , \quad z = z(t)$$

of a real variable t linked by the following system of differential equations:

$$\begin{aligned} x' &= -9x + 8y \\ y' &= -12x + 11y \\ z' &= -8x + 6y + z \end{aligned}$$

Find the general solution to this system.

(h) Find the particular solution to the system described in the previous part given that $x(0) = 1$, $y(0) = 3$ and $z(0) = 0$.

2. Consider the subset

$$W = \{(x, y, z, w) \in \mathbb{R}^4 \mid 2x + y - 2z + w = 0\}$$

of the real inner product space $\mathbb{R}^4$, and let $\mathbf{v} = (4, 3, -2, 3)$.

(a) What is meant by a *subspace* of a vector space? Verify that W is a subspace of $\mathbb{R}^4$.

(b) Verify that $\mathbf{v}_1 = (-1, 2, 0, 0)$, $\mathbf{v}_2 = (1, 0, 1, 0)$ and $\mathbf{v}_3 = (-1, 0, 0, 2)$ span W.

(c) What is the *dimension* of a vector space? Explain why W is three dimensional.

(d) What is an *orthonormal basis* for an inner product space?

(e) Apply the Gram-Schmidt process to obtain an orthonormal basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ for W.

(f) Use the orthonormal basis from the previous part to find the projection $\text{proj}_W \mathbf{v}$ of the vector $\mathbf{v}$ onto W.

(g) Find the shortest distance from the point $(4, 3, -2, 3)$ in $\mathbb{R}^4$ to the hyperplane W.

3. This exercise illustrates an alternative method for finding the closest point on a hyperplane W to a point $\mathbf{v}$ in $\mathbb{R}^n$ and the shortest distance.

(a) Use the coefficients in the equation defining W to find a vector $\mathbf{n}$ orthogonal to every vector in W, where

$$W = \{(x, y, z, w) \in \mathbb{R}^4 \mid 2x + y - 2z + w = 0\} .$$

(b) Find the projection of $\mathbf{v} = (4, 3, -2, 3)$ in the direction of $\mathbf{n}$, and the length of this projection. (The latter should coincide with the answer to part (g) of the previous exercise.)

(c) Let $\mathcal{L}$ be the line in $\mathbb{R}^4$ containing the point $\mathbf{v}$ and parallel to $\mathbf{n}$. The collection of points $\mathbf{r}$ that lie on the line $\mathcal{L}$ are represented by the equation

$$\mathbf{r} = \mathbf{v} + \lambda \mathbf{n} .$$

where $\lambda \in \mathbb{R}$. Find the value of λ such that $\mathbf{r} \in W$. (The magnitude of $\lambda \mathbf{n}$ should coincide with the answer to the previous part.)

(d) Use the answer to either of the previous parts to find the closest point on W to $\mathbf{v}$. (This should coincide with the answer to part (f) of the previous exercise.)

(e) Use any method to find the equation in variables x, y, z, w that describes the subspace W' of $\mathbb{R}^4$ spanned by

$$\mathbf{v}_1 = (1, 1, 1, -1) , \quad \mathbf{v}_2 = (0, 1, 1, 0) , \quad \mathbf{v}_3 = (1, 0, 1, 2)$$

(f) Let W' be the subspace (hyperplane) defined in part (e). Use the method of parts (a),(b),(c),(d) to find the closest point on W' to $\mathbf{v}' = (4, 0, 3, -2)$ and the shortest distance from $\mathbf{v}'$ to W'.

4. (a) What is a *probability vector* and what is a *stochastic matrix*? What further property does a stochastic matrix need to be called *regular*?

(b) Verify that if S is a stochastic $n \times n$ matrix and $\mathbf{v}$ is a probability vector with n entries then $S\mathbf{v}$ is also a probability vector. Deduce that the set of stochastic matrices of the same size is closed under matrix multiplication.

(c) Consider the matrix

$$S = \begin{bmatrix} 0.4 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.3 \\ 0.3 & 0.6 & 0.4 \end{bmatrix} .$$

(i) Explain briefly why S is regular stochastic.

(ii) Row reduce $I - S$ to find a steady state probability vector for S.

(iii) Write down $\lim_{n\to\infty} S^n$, as predicted by the general theory.

(iv) Find an invertible matrix P and a diagonal matrix D such that $S = PDP^{-1}$ and use this to verify your answer to part (iii).

Hints and short solutions:

1. (a) $\chi(\lambda) = \lambda^3 - 3\lambda^2 - \lambda + 3 \ = \ (\lambda-1)(\lambda+1)(\lambda-3)$

(b) $\det(M) \ = \ -\chi(0) \ = \ -3 \neq 0$

(c) $M^{-1} = -\frac{1}{3}(M^2 - 3M - I) = -\frac{1}{3}\begin{bmatrix} 11 & -8 & 0 \\ 12 & -9 & 0 \\ 16 & -10 & -3 \end{bmatrix}$

(d) $M^{-1} = \begin{bmatrix} 5 & 7 & 0 \\ 9 & 3 & 0 \\ 12 & 12 & 1 \end{bmatrix}$

(e) $1, \left\{ \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, -1, \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, 3, \left\{ \begin{bmatrix} 2t \\ 3t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

(f) $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(g) $x = Be^{-t} + 2Ce^{3t}\,,\ y = Be^{-t} + 3Ce^{3t}\,,\ z = Ae^{t} + Be^{-t} + Ce^{3t}$

(h) $x = -3e^{-t} + 4e^{3t}\,,\ y = -3e^{-t} + 6e^{3t}\,,\ z = e^{t} - 3e^{-t} + 2e^{3t}$

2. (a) nonempty set closed under vector addition and scalar multiplication.

(b) rewrite W so that typical entry is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

(c) size of any basis; check directly that elements of B are linearly independent

(d) basis in which all vectors have length 1 and any two distinct vectors are orthogonal

(e) $\left\{ \frac{1}{\sqrt{5}}(-1,2,0,0)\,,\ \frac{1}{3\sqrt{5}}(4,2,5,0)\,,\ \frac{1}{3\sqrt{10}}(-2,-1,2,9) \right\}$ (f) $\frac{2}{5}(1,3,4,3)$ (g) $\frac{9\sqrt{10}}{5}$

3. (a) $(2,1,-2,1)$ (b) $\frac{9}{5}(2,1,-2,1)$, $\frac{9\sqrt{10}}{5}$ (c) $\frac{9}{5}$ (d) $\frac{2}{5}(1,3,4,3)$

(e) $x + 3y - 3z + w = 0$ (f) $\frac{7\sqrt{5}}{10}$, $\frac{3}{20}(29,7,13,-11)$

4. (a) all entries nonnegative and add up to 1; all columns probability vectors; all entries positive in some positive power

(b) interchange summations in a double summation

(c) (ii) $\begin{bmatrix} \frac{10}{33} \\ \frac{9}{33} \\ \frac{14}{33} \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{10}{33} & \frac{10}{33} & \frac{10}{33} \\ \frac{9}{33} & \frac{9}{33} & \frac{9}{33} \\ \frac{14}{33} & \frac{14}{33} & \frac{14}{33} \end{bmatrix}$ (iv) $\begin{bmatrix} 10 & -1 & -1 \\ 9 & 0 & -2 \\ 14 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & -\frac{1}{10} \end{bmatrix}$