MATH 2022 Linear and Abstract Algebra

LECTURE 11 Wednesday 20/03/2019

Practice Quiz (continued):

Q17/ Put $\alpha = (13)(2465)$, $\beta = (1425)(63)$.

Find $\gamma = \beta^{-1}\alpha\beta$.

Solution: $\alpha = (1\,3)(2\,4\,6\,5)$

$$\alpha^{\beta} = \beta^{-1}\alpha\beta = (4\,6)(5\,2\,3\,1)$$

$$= (4\,6)(1\,5\,2\,3)$$

answer on practice quiz

Q18/ Put $\alpha = (1\,3)(4\,2\,5\,6)$, $\gamma = (4\,5)(1\,3\,2\,6)$.

Find β such that $\beta^{-1}\alpha\beta = \gamma$.

Solution: Here is an exhaustive list of possibilities:

$\alpha = (1\,3)(4\,2\,5\,6)$	β
$\gamma = (4\,5)(1\,3\,2\,6)$	$(1\,4)(2\,3\,5)$
$= (5\,4)(1\,3\,2\,6)$	$(1\,5\,2\,3\,4)$
$= (4\,5)(3\,2\,6\,1)$	$(1\,4\,3\,5\,6)$
$= (5\,4)(3\,2\,6\,1)$	$(1\,5\,6)(3\,4)$
$= (4\,5)(2\,6\,1\,3)$	$(1\,4\,2\,6\,3\,5)$

$\alpha = (1\,3)(4\,2\,5\,6)$	β
$\gamma = (4\,5)(1\,3\,2\,6)$	$(1\,4)(2\,3\,5)$
$= (5\,4)(1\,3\,2\,6)$	$(1\,5\,2\,3\,4)$
$= (4\,5)(3\,2\,6\,1)$	$(1\,4\,3\,5\,6)$
$= (5\,4)(3\,2\,6\,1)$	$(1\,5\,6)(3\,4)$
$= (4\,5)(2\,6\,1\,3)$	$(1\,4\,2\,6\,3\,5)$
$= (5\,4)(2\,6\,1\,3)$	$(1\,5)(2\,6\,3\,4)$
$= (4\,5)(6\,1\,3\,2)$	$(1\,4\,6\,2)(3\,5)$
$= (5\,4)(6\,1\,3\,2)$	$(1\,5\,3\,4\,6\,2)$

answer on practice sheet

Q20/ Which of the following can be reached from the 8-puzzle

1	2	3
4	5	6
7	8	

?

(a)

2	4	
8	1	5
3	6	7

possible iff

2	4	5
8	1	7
3	6	

possible

(1 5 3 7 6 8 4 2)

odd ✗ impossible

Q20/ Which of the following can be reached from the 8-puzzle

1	2	3
4	5	6
7	8	9

?

(a)

2	4	9
8	1	5
3	6	7

Alternatively: (1 5 6 8 4 2)(3 7 9) odd,

and blank an even number of interchanges away from bottom corner, so impossible X

Q20/ Which of the following can be reached from the 8-puzzle

1	2	3
4	5	6
7	8	

?

(b)

7	2	4
6		8
1	5	3

possible iff

7	2	4
6	8	3
1	5	

possible

(1 7)(3 6 4)(5 8)

even ✓ possible

Q20/ Which of the following can be reached from the 8-puzzle

1	2	3
4	5	6
7	8	9

?

(b)

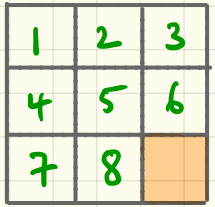
7	2	4
6	9	8
1	5	3

Alternatively : (1 7)(3 9 5 8 6 4) even,

and blank on even number of interchanges away

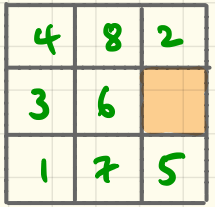
from bottom corner, so possible ✓

Q20/ Which of the following can be reached from the 8-puzzle

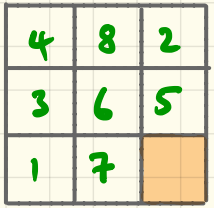


?

(c)



possible iff



possible

(1 7 8 2 3 4)(5 6)

even ✓ possible

Q20/ Which of the following can be reached from the 8-puzzle

1	2	3
4	5	6
7	8	9

?

(c)

4	8	2
3	6	9
1	7	5

Alternatively: (1 7 8 2 3 4)(5 9 6) odd,

and blank an odd number of transitions away

from bottom corner, so possible ✓

Q20/ Which of the following can be reached from the 8-puzzle

1	2	3
4	5	6
7	8	

?

(d)

8	3	4
2		7
6	5	1

(1 9 5 8)(2 4 3)(6 7) even,

and blank an <u>even</u> number of interchanges away from bottom corner, so possible ✓

Q20/ Which of the following can be reached from the 8-puzzle

1	2	3
4	5	6
7	8	

?

(e)

	7	1
6	4	2
3	5	8

(1 3 7 2 6 4 5 8 9) even

and blank an even number of interchanges away from bottom corner, so possible ✓

Q10/ Let $\theta \in \mathbb{R}$ and put

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad T_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Which one of the following is true?

(a) $R_{\pi/3}^3 = I = T_{\pi/2}^2$

(b) $R_{2\pi/3}^3 = I = T_{2\pi/3}^3$

(c) $R_{\pi/4}^4 = I = T_{\pi/4}^4$

(d) $R_{\pi/2} T_{2\pi/3} R_{\pi/2} = T_{4\pi/3}$

(e) $T_{\pi/2} R_{2\pi/3} T_{\pi/2} = R_{4\pi/3}$

Q10/ Let $\theta \in \mathbb{R}$ and put

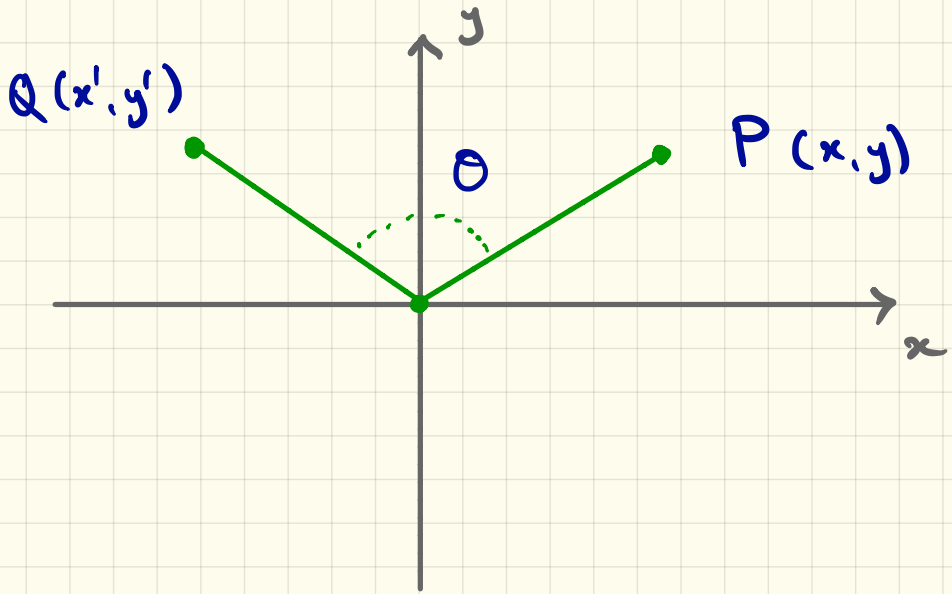
$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad T_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

R_θ: **rotation** matrix

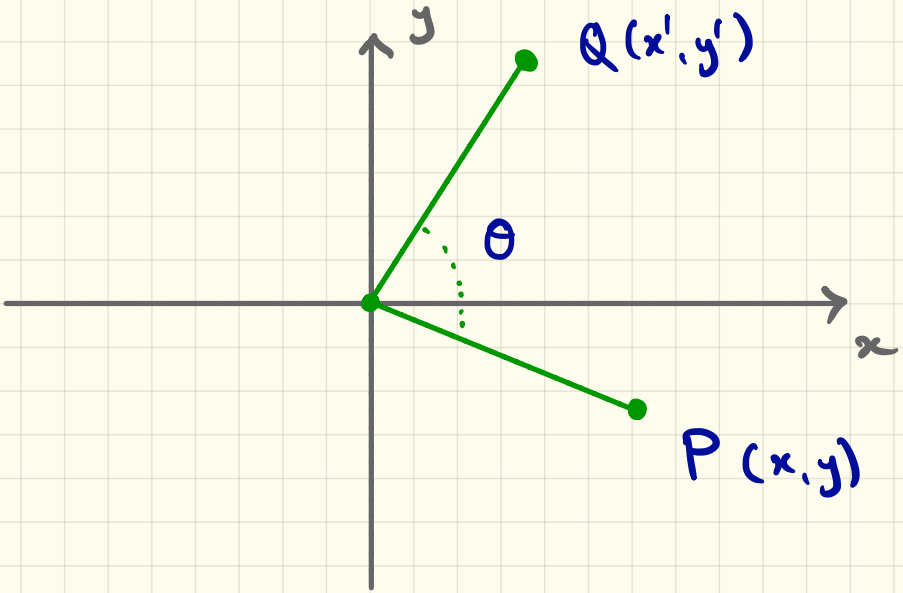
T_θ: **reflection** matrix

Facts: (1) $R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ if the point (x, y) is rotated into the point (x', y') in the xy-plane by rotating θ radians anticlockwise about the origin.

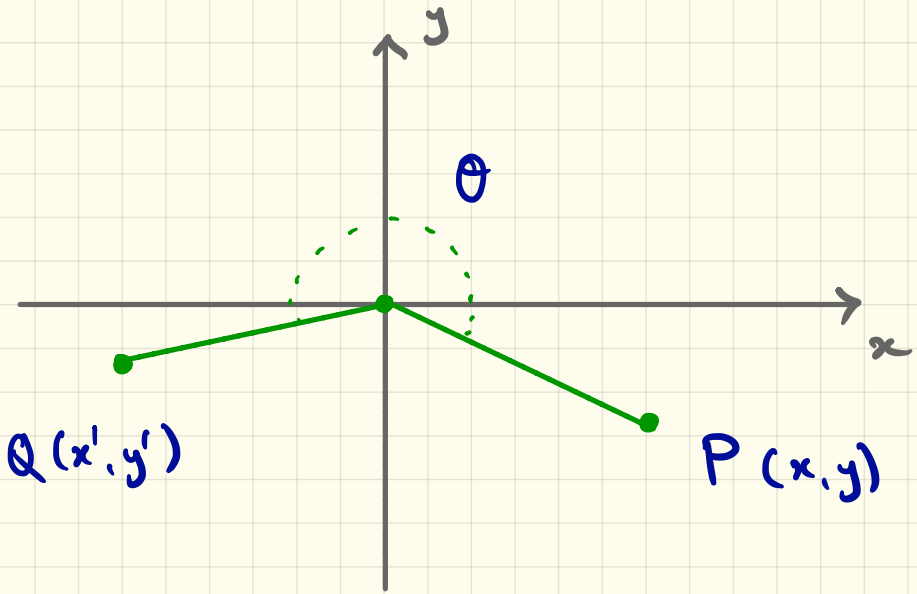
Facts: (1) $R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ if the point (x, y) is rotated into the point (x', y') in the xy-plane by rotating θ radians anticlockwise about the origin.



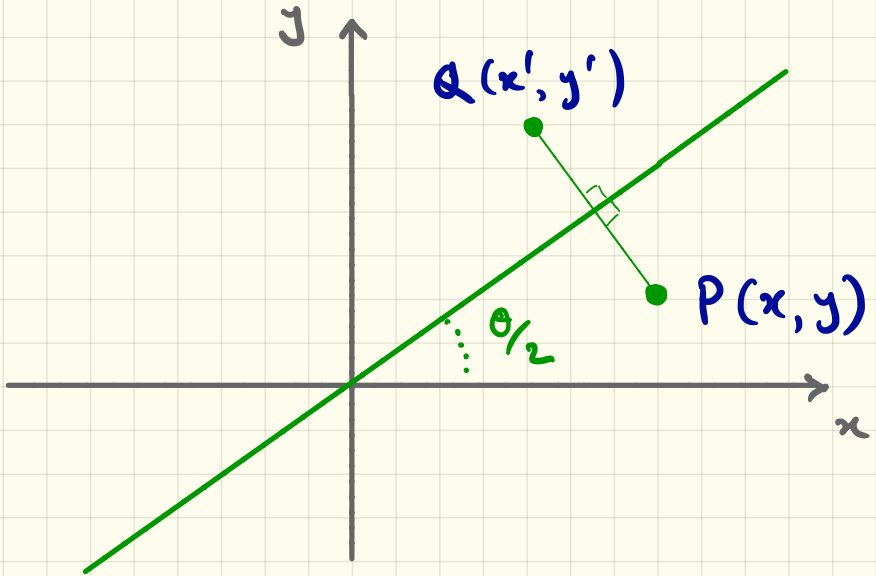
Facts: (1) $R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ if the point (x, y) is rotated into the point (x', y') in the xy-plane by rotating θ radians anticlockwise about the origin.



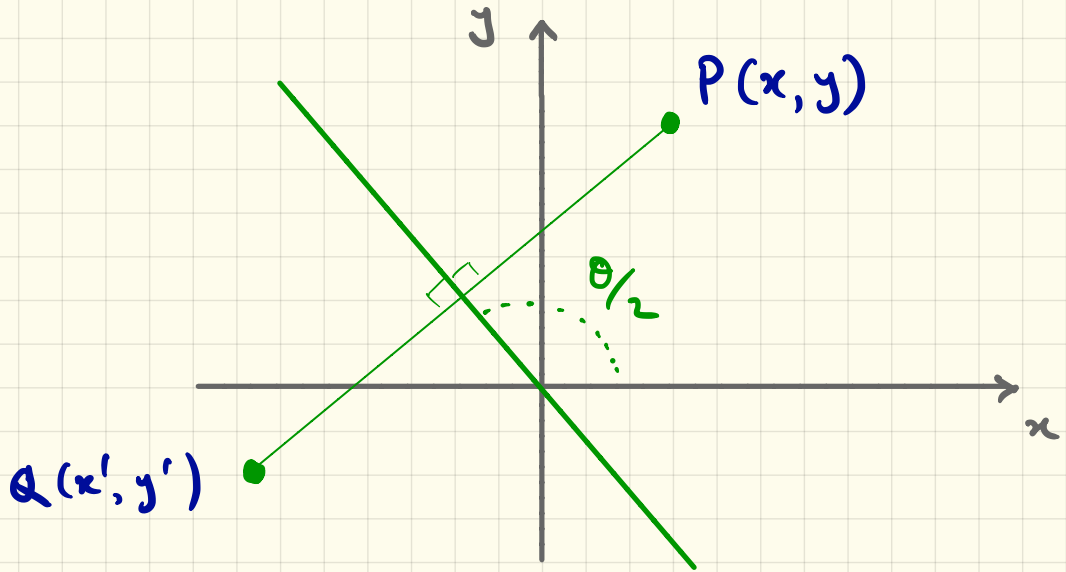
Facts: (1) $R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ if the point (x, y) is rotated into the point (x', y') in the xy-plane by rotating θ radians anticlockwise about the origin.



Facts: (2) $T_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ if the point (x, y) is reflected to the point (x', y') in the xy-plane through the line passing through the origin making an angle $\theta/2$ with the positive x-axis.



Facts: (2) $T_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ if the point (x, y) is reflected to the point (x', y') in the xy-plane through the line passing through the origin making an angle $\theta/2$ with the positive x-axis.



Facts: (3) $R_{\theta_1} R_{\theta_2} = R_{\theta_1+\theta_2}$

(4) $R_{2\pi} = \begin{bmatrix} \cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e. $R_{2\pi} = I$ (identity matrix)

e.g. $R_{\pi/3} R_{\pi/6} = R_{(\pi/3+\pi/6)} = R_{\pi/2}$

i.e. $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ✓ (check)

Facts: (3) $R_{\theta_1} R_{\theta_2} = R_{\theta_1 + \theta_2}$

(4) $R_{2\pi} = \begin{bmatrix} \cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e. $R_{2\pi} = I$ (identity matrix)

$R_\theta^n = R_{n\theta}$ for any $\theta \in \mathbb{R}$, $n \in \mathbb{Z}$

e.g. $R_{\pi/2}^4 = R_{2\pi} = I$, $R_{2\pi/3}^3 = R_{2\pi} = I$.

Facts: (3) $R_{\theta_1} R_{\theta_2} = R_{\theta_1 + \theta_2}$

(4) $R_{2\pi} = \begin{bmatrix} \cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e. $R_{2\pi} = I$ (identity matrix)

$R_\theta R_{-\theta} = R_{(\theta - \theta)} = R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

i.e. $R_\theta^{-1} = R_{-\theta}$

Facts:

(3) $R_{\theta_1} R_{\theta_2} = R_{\theta_1 + \theta_2}$

(4) $R_{2\pi} = R_0 = I$

(5) $R_\theta^n = R_{n\theta}$, $R_\theta^{-1} = R_{-\theta}$

(6) $T_\theta^2 = I$, $T_\theta^{-1} = T_\theta$

self-inverse

(7) $\boxed{T R T = R^{-1}}$

for any rotation R and any reflection T.

Facts:

$$TRT = R^{-1}$$

for any rotation R and any reflection T.

"Conjugating a rotation by a reflection inverts the rotation"

$TRT = R^{-1}$

$\Leftrightarrow$ $TRTR = I$

$\Leftrightarrow$ $T(TRTR) = TI = T$

$\Leftrightarrow$ $(TT)RTR = T$

$\Leftrightarrow$ $RTR = T$

Facts:

$$TRT = R^{-1}$$

for any rotation R and any reflection T.

"Conjugating a rotation by a reflection inverts the rotation"

$$RTR = T$$

"A reflection is unchanged by premultiplying and postmultiplying by the same rotation"

Q10/ Let $\theta \in \mathbb{R}$ and put

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad T_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Which one of the following is true?

✗ (a) $R_{\pi/3}^3 \overset{\times}{=} I \overset{\checkmark}{=} T_{\pi/2}^2$ ✗ (b) $R_{2\pi/3}^3 \overset{\checkmark}{=} I \overset{\times}{=} T_{2\pi/3}^3$

✗ (c) $R_{\pi/4}^4 \overset{\times}{=} I \overset{\checkmark}{=} T_{\pi/4}^4$ ✗ (d) $R_{\pi/2} T_{2\pi/3} R_{\pi/2} \overset{\times}{=} T_{4\pi/3}$

✓ (e) $T_{\pi/2} R_{2\pi/3} T_{\pi/2} \overset{\checkmark}{=} R_{4\pi/3} = R_{2\pi/3}^{-1}$