MATH 2022 Linear and Abstract Algebra

LECTURE 22 Tuesday 16/04/2019

Vector spaces

- very abstract general arithmetics

  intertwined with field arithmetic

- appear throughout mathematics & applications

- capture with great precision & power the intention of

The Linear Algebra Principle:

Arrange things to move "in straight lines"

The abstract definition:

Throughout, let F denote a fixed field

(e.g. $F = \mathbb{R}, \mathbb{C}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, \ldots$),

elements of which are called **scalars**.

A **vector space** V (**over a field** F) is an abelian group with respect to addition, with a **compatible scalar multiplication**:

$$\underset{\sim}{v}, \underset{\sim}{w} \in V \Rightarrow \underset{\sim}{v} + \underset{\sim}{w} \in V$$

vector sum

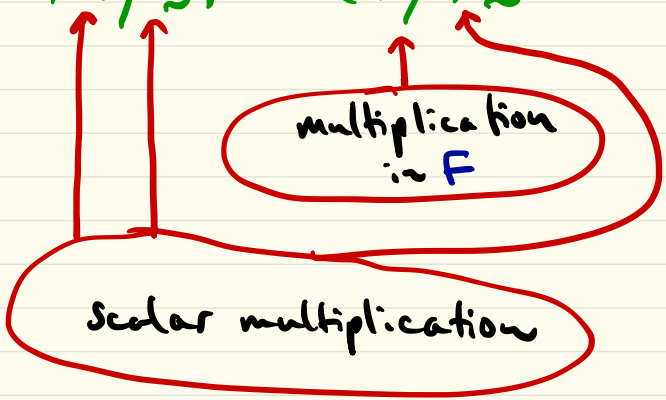
$$\underset{\sim}{v} \in V, \lambda \in F \Rightarrow \lambda \underset{\sim}{v} \in V$$

scalar product

We require

(1) associativity of scalar multiplication:

$$(\forall \underset{\sim}{v} \in V)(\forall \lambda, \mu \in F) \quad \lambda(\mu \underset{\sim}{v}) = (\lambda\mu)\underset{\sim}{v}$$



(2) right distributivity:

$$(\forall \underset{\sim}{v} \in V)(\forall \lambda, \mu \in F) \quad (\lambda + \mu)\underset{\sim}{v} = \lambda\underset{\sim}{v} + \mu\underset{\sim}{v}$$

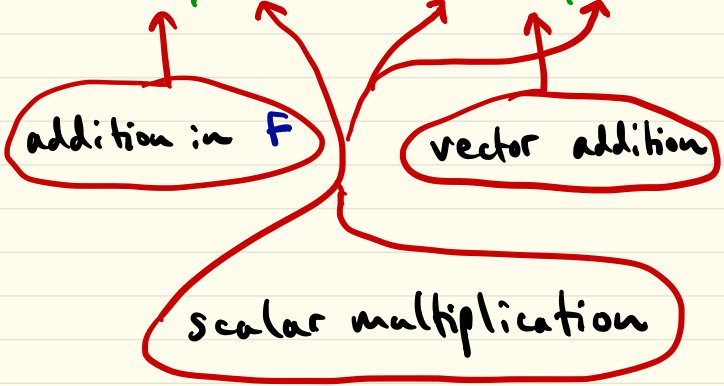
We require

(1) associativity of scalar multiplication:

$$(\forall \underset{\sim}{v} \in V)(\forall \lambda, \mu \in F) \quad \lambda(\mu \underset{\sim}{v}) = (\lambda\mu)\underset{\sim}{v}$$

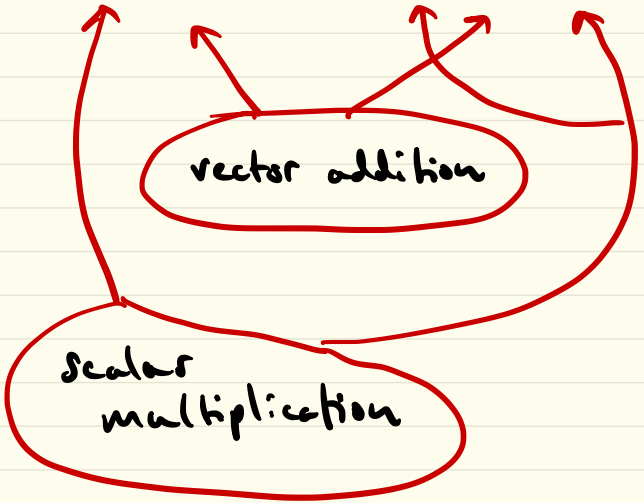
(2) right distributivity:

$$(\forall \underset{\sim}{v} \in V)(\forall \lambda, \mu \in F) \quad (\lambda+\mu)\underset{\sim}{v} = \lambda\underset{\sim}{v} + \mu\underset{\sim}{v}$$



(3) left distributivity :

$(\forall \underset{\sim}{v}, \underset{\sim}{w} \in V)(\forall \lambda \in F)$ $\lambda(\underset{\sim}{v}+\underset{\sim}{w}) = \lambda\underset{\sim}{v} + \lambda\underset{\sim}{w}$



(4) scalar identity element : $(\forall \underset{\sim}{v} \in V)\ 1\underset{\sim}{v} = \underset{\sim}{v}$

We require

(1) associativity of scalar multiplication:

$$(\forall \underset{\sim}{v} \in V)(\forall \lambda, \mu \in F) \quad \lambda(\mu \underset{\sim}{v}) = (\lambda\mu)\underset{\sim}{v}$$

(2) right distributivity:

$$(\forall \underset{\sim}{v} \in V)(\forall \lambda, \mu \in F) \quad (\lambda+\mu)\underset{\sim}{v} = \lambda\underset{\sim}{v} + \mu\underset{\sim}{v}$$

(3) left distributivity:

$$(\forall \underset{\sim}{v}, \underset{\sim}{w} \in V)(\forall \lambda \in F) \quad \lambda(\underset{\sim}{v}+\underset{\sim}{w}) = \lambda\underset{\sim}{v} + \lambda\underset{\sim}{w}$$

(4) scalar identity element: $(\forall \underset{\sim}{v} \in V) \quad 1\underset{\sim}{v} = \underset{\sim}{v}$

For $(V,+)$ to be an abelian group means

(5) $(\forall \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V) \quad \underset{\sim}{u}+(\underset{\sim}{v}+\underset{\sim}{w}) = (\underset{\sim}{u}+\underset{\sim}{v})+\underset{\sim}{w}$

(associativity of addition)

(6) $(\forall \underset{\sim}{v}, \underset{\sim}{w} \in V) \quad \underset{\sim}{v}+\underset{\sim}{w} = \underset{\sim}{w}+\underset{\sim}{v}$

(commutativity of addition)

(7) $(\exists \underset{\sim}{0} \in V)(\forall \underset{\sim}{v} \in V) \quad \underset{\sim}{v}+\underset{\sim}{0} = \underset{\sim}{0}+\underset{\sim}{v} = \underset{\sim}{v}$

(existence of zero)

(8) $(\forall \underset{\sim}{v} \in V)(\exists \underset{\sim}{w} \in V) \quad \underset{\sim}{v}+\underset{\sim}{w} = \underset{\sim}{w}+\underset{\sim}{v} = 0$ $\underset{\sim}{w} = -\underset{\sim}{v}$

(existence of negatives)

Note, the zero vector $\underset{\sim}{0}$ is unique, for if $\underset{\sim}{0}$ and $\underset{\sim}{0}'$ both satisfy (7) then

$$\underset{\sim}{0} = \underset{\sim}{0} + \underset{\sim}{0}' = \underset{\sim}{0}'.$$

The negative of a vector is also unique, for if

$$\underset{\sim}{v} + \underset{\sim}{w} = \underset{\sim}{w} + \underset{\sim}{v} = \underset{\sim}{0} = \underset{\sim}{v} + \underset{\sim}{w}' = \underset{\sim}{w}' + \underset{\sim}{v}$$

then

$$\underset{\sim}{w} = \underset{\sim}{w} + \underset{\sim}{0} = \underset{\sim}{w} + (\underset{\sim}{v} + \underset{\sim}{w}') = (\underset{\sim}{w} + \underset{\sim}{v}) + \underset{\sim}{w}' = \underset{\sim}{0} + \underset{\sim}{w}' = \underset{\sim}{w}'.$$

We denote the negative of $\underset{\sim}{v}$ by $-\underset{\sim}{v}$,

and this enables us to define subtraction in V:

We denote the negative of $\underset{\sim}{v}$ by $-\underset{\sim}{v}$,

and this enables us to define subtraction in V:

$$(\forall\, \underset{\sim}{v}, \underset{\sim}{w} \in V) \quad \underset{\sim}{v} - \underset{\sim}{w} \overset{\text{def}}{=} \underset{\sim}{v} + (-\underset{\sim}{w})$$

and then we get an analogue of left distributivity:

$$(\forall\, \underset{\sim}{v}, \underset{\sim}{w} \in V)(\forall \lambda \in F) \quad \lambda(\underset{\sim}{v} - \underset{\sim}{w}) = \lambda(\underset{\sim}{v} + (-\underset{\sim}{w}))$$

$$= \lambda \underset{\sim}{v} + \lambda(-\underset{\sim}{w})$$

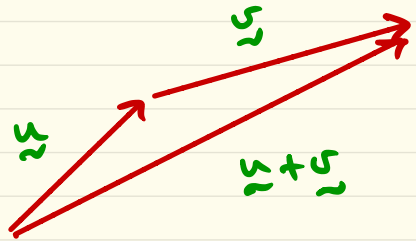
$$= \lambda \underset{\sim}{v} + (-(\lambda \underset{\sim}{w}))$$

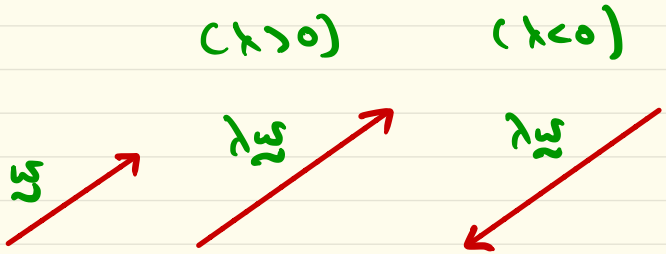
$$= \lambda \underset{\sim}{v} - \lambda \underset{\sim}{w}$$

exercise to verify this

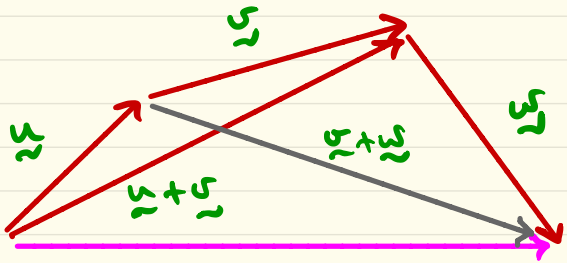
Examples of vector spaces:

(1) Let $F = \mathbb{R}$ and $V = \{$ directed line segments $\}$ in the plane or in space





e.g.



demonstrates associativity of $+$

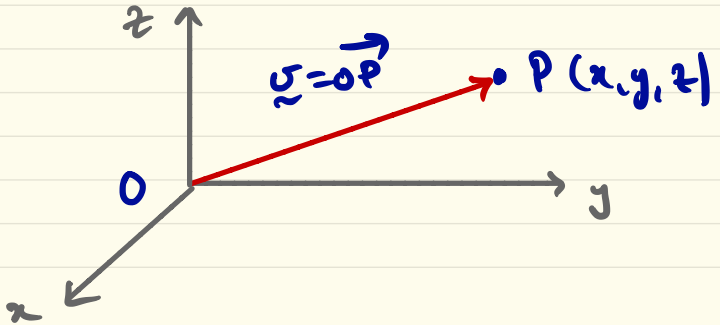
$$(\underset{\sim}{u}+\underset{\sim}{v})+\underset{\sim}{w} = \underset{\sim}{u}+(\underset{\sim}{v}+\underset{\sim}{w})$$

(2) $V = F^n = \{(a_1, \dots, a_n) \mid a_1, \dots, a_n \in F\}$

with coordinatewise addition & scalar multiplication.

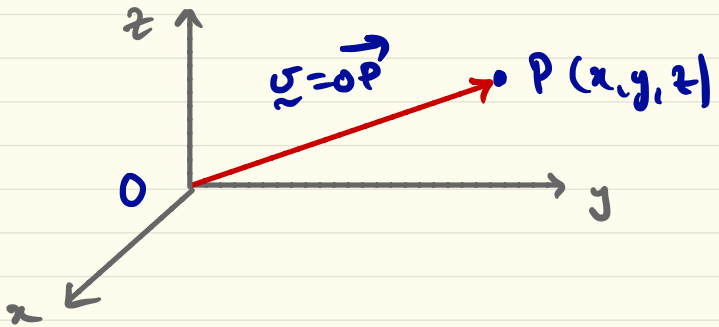
routine to check all the vector space axioms hold ✓

e.g. $\mathbb{R}^3$ is typically identified with the vector space of directed line segments in space, using coordinates of position vectors, fixed at the origin:



$\underset{\sim}{v} \equiv (x, y, z)$

e.g. $\mathbb{R}^3$ is typically identified with the vector space of directed line segments in space, using coordinates of position vectors, fixed at the origin:



$\underset{\sim}{v} \equiv (x, y, z)$

more formally: $\underset{\sim}{v} \mapsto (x, y, z)$

is a vector space isomorphism

(3) Put $\mathrm{Mat}_{m,n}(F) = \{ m \times n \text{ matrices over } F \}$.

– forms a vector space with respect to matrix addition & scalar multiplication.

Note

$\mathrm{Mat}_{1,n}(F) = \{ \text{row vectors of length } n \}$,

$\mathrm{Mat}_{m,1}(F) = \{ \text{column vectors of length } m \}$.

e.g.

$$\mathrm{Mat}_{3,1}(\mathbb{R}) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\},$$

$$\mathrm{Mat}_{1,3}(\mathbb{R}) = \{ \begin{bmatrix} x & y & z \end{bmatrix} \mid x, y, z \in \mathbb{R} \}.$$

We identify n-tuples with row vectors:

$$(x_1, \dots, x_n) \equiv [x_1 \ \dots \ x_n],$$

so that

$$\boxed{F^n \equiv \mathrm{Mat}_{1,n}(F)}.$$

(Some authors, e.g. Poole, identify F^n with $\mathrm{Mat}_{n,1}(F)$, column vectors

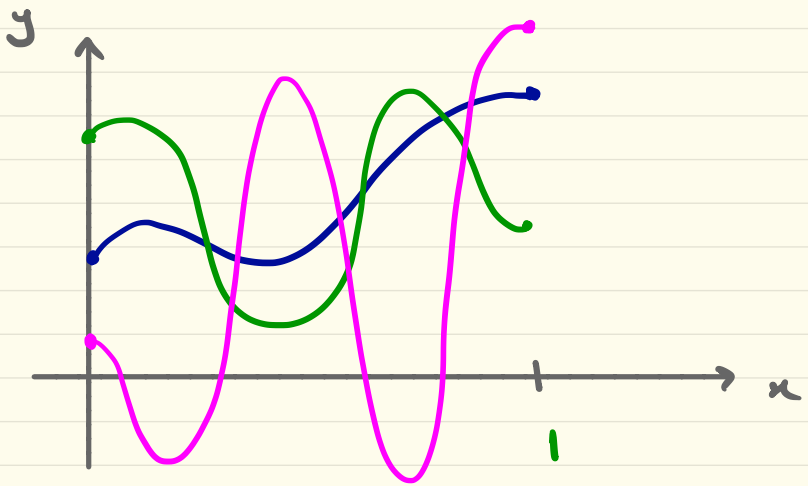
though we prefer to avoid this, because of possible confusion when combining vectors with matrix multiplication.)

(4) Let F be a field and X any set.

Put

$$F^X = \{ \text{functions} : X \to F \}.$$

e.g. $$\mathbb{R}^{[0,1]} = \{ \text{functions} : [0,1] \to \mathbb{R} \}$$



Then F^X forms a vector space under pointwise operations:

if $f, g \in F^X$ and $\lambda \in F$ then define

$$f+g,\ \lambda f \in F^X$$

by

$$\left.\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (\lambda f)(x) &= \lambda f(x) \end{aligned}\right\} \text{ for all } x \in X.$$

The zero vector in this context is the zero function

$$\underset{\sim}{0} : X \to F,\ x \mapsto 0 \quad \forall x \in X.$$

Then F^X forms a vector space under pointwise operations:

if $f, g \in F^X$ and $\lambda \in F$ then define

$$f+g,\ \lambda f \in F^X$$

by

$$\left.\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (\lambda f)(x) &= \lambda f(x) \end{aligned}\right\} \text{ for all } x \in X.$$

The zero vector in this context is the zero function

$$\underset{\sim}{0} : X \to F,\ x \mapsto 0 \quad \forall x \in X.$$

The negative of $f \in F^X$ is $-f$ where

$$-f : X \to F,\ x \mapsto -f(x) \quad \forall x \in X.$$

(5) We may identify F^n with

$$F^{\{1,\dots,n\}} = \{ \text{functions} : \{1,\dots,n\} \to F \}$$

where an n-tuple $(a_1,\dots,a_n) \in F^n$ may be identified with a function

$$s : \{1,\dots,n\} \to F$$

where

$$\begin{aligned} s : \quad 1 &\longmapsto a_1 \\ 2 &\longmapsto a_2 \\ &\vdots \\ n &\longmapsto a_n \end{aligned}$$

sequence thought of as a function of the subscripts

(6) Let $V = \mathbb{R}^{\mathbb{R}} = \{ \text{functions} : \mathbb{R} \to \mathbb{R} \}$

Then V contains many important subsets which are themselves vector spaces,

e.g. $\mathcal{C}(\mathbb{R}) = \{ \text{continuous functions} : \mathbb{R} \to \mathbb{R} \}$,

$\mathcal{C}_1(\mathbb{R}) = \{ \text{differentiable} \quad " \quad : \mathbb{R} \to \mathbb{R} \}$,

$\mathcal{C}_2(\mathbb{R}) = \{ \text{twice differentiable} \quad " \quad : \mathbb{R} \to \mathbb{R} \}$,

$\mathcal{C}_\infty(\mathbb{R}) = \{ \text{infinitely differentiable} \quad " \quad : \mathbb{R} \to \mathbb{R} \}$.

examples of subspaces of $\mathbb{R}^{\mathbb{R}}$ (see later)

(7) For each $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ put

$$\mathbb{P}_n = \{ a_0 + a_1 x + \cdots + a_n x^n \mid a_0, a_1, \dots, a_n \in F \}$$

the vector space of polynomials of degree $\leq n$ over F.

Then

$$\mathbb{P}_0 = \{ a_0 \mid a_0 \in F \} \equiv F$$

(constant polynomials)

$$\mathbb{P}_1 = \{ a_0 + a_1 x \mid a_0, a_1 \in F \}$$

(linear polynomials)

$$\mathbb{P}_2 = \{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in F \}$$

(quadratic polynomials)

$$P_3 = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \in F \}$$

(cubic polynomials)

etc

Put $$P = \bigcup_{n \in \mathbb{N}} P_n = \{ \text{all polynomials over } F \}$$

Then

$$P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq P_3 \subsetneq \cdots \subsetneq P_n \subsetneq \cdots \subsetneq P$$

- cascade of subspaces with proper set containment at each step.

(8) Let

$$V = \{ a_0 + a_1 x + \cdots + a_n x^n + \cdots \mid a_0, a_1, \ldots, a_n, \ldots \in F \}$$

the vector space of formal power series

under usual addition and scalar multiplication of power series.

Then

$$P \subsetneq V$$

space of polynomials (finite power series)