MATH 2022 Linear and Abstract Algebra

LECTURE 26 Wednesday 01/05/2019

Basis and dimension

- a basis for a vector space provides a foundation for accessing all vectors
- bases (plural of basis) of vector spaces facilitate matrix arithmetic
- choosing the right bases or modifying bases lead to simplified calculations & solutions
- dimension is the size of a basis of a vector space and captures precisely notions of "degrees of freedom"

Let V be a vector space over a field F.

A subset B of V is called a basis for V if

(i) B is a spanning set for V, that is,

$$V = \langle B \rangle = \{ \text{all possible linear combinations of elements of } B \}$$

So every $\underset{\sim}{v} \in V$ can be expressed as a linear combination $\underset{\sim}{v} = \lambda_1 \underset{\sim}{b}_1 + \cdots + \lambda_n \underset{\sim}{b}_n$ for some $\underset{\sim}{b}_1, \ldots, \underset{\sim}{b}_n \in B$ and $\lambda_1, \ldots, \lambda_n \in F$

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i.e.

$$\boxed{\begin{array}{l} \text{for all distinct } \underset{\sim}{b}_1, \ldots, \underset{\sim}{b}_n \in B, \text{ we have} \\ (\forall \lambda_1, \ldots, \lambda_n \in F)\ \lambda_1 \underset{\sim}{b}_1 + \cdots + \lambda_n \underset{\sim}{b}_n = \underset{\sim}{0} \Rightarrow \lambda_1 = \cdots = \lambda_n = 0. \end{array}}$$

Note: this definition allows B to be infinite, though in most examples & applications, B will be finite.

Two ingredients of a basis:

(i) the spanning property guarantees that one can reach everything in V by applying vector space operations to elements of B.

(ii) LI of B guarantees that no element of B is superfluous.

Examples (standard bases):

$\{(1,0), (0,1)\}$ is a basis for F^2,

$\{(1,0,0), (0,1,0), (0,0,1)\}$ " " " " F^3.

More generally, $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_n\}$ is a basis for F^n

where

$$\underset{\sim}{e}_i = (0, \ldots, 0, \underset{\text{i th place}}{\underset{\uparrow}{1}}, 0, \ldots 0) \quad \text{for } i = 1, \ldots, n,$$

called the <u>standard basis</u> for F^n.

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Note, if $\underset{\sim}{v} = (\lambda_1, \lambda_2, \dots, \lambda_n) \in F^n$ then

$$\underset{\sim}{v} = \lambda_1 \underset{\sim}{e}_1 + \lambda_2 \underset{\sim}{e}_2 + \dots + \lambda_n \underset{\sim}{e}_n,$$

so that the coordinates of $\underset{\sim}{v}$ become coefficients of the basis elements.

$\boxed{\begin{array}{l}\underline{\text{General fact}}: \text{ Let } B = \{\underset{\sim}{b}_1, \ldots, \underset{\sim}{b}_n\} \text{ be a basis for a vector space } V. \text{ Then} \\ (\forall \underset{\sim}{v} \in V)(\exists \ \underline{\underline{\text{unique}}} \ \lambda_1, \ldots, \lambda_n) \\ \underset{\sim}{v} = \lambda_1 \underset{\sim}{b}_1 + \cdots + \lambda_n \underset{\sim}{b}_n .\end{array}}$

<u>Proof</u>: That such linear combinations exist follows from the spanning property of B.

To prove uniqueness, suppose

$$\underset{\sim}{v} = \lambda_1 \underset{\sim}{b}_1 + \cdots + \lambda_n \underset{\sim}{b}_n = \mu_1 \underset{\sim}{b}_1 + \cdots + \mu_n \underset{\sim}{b}_n$$

for some $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n \in F$.

Then

$$\underset{\sim}{0} = \underset{\sim}{v} - \underset{\sim}{v}$$

$$= (\lambda_1 \underset{\sim}{b}_1 + \dots + \lambda_n \underset{\sim}{b}_n) - (\mu_1 \underset{\sim}{b}_1 + \dots + \mu_n \underset{\sim}{b})$$

$$= (\lambda_1 - \mu_1) \underset{\sim}{b}_1 + \dots + (\lambda_n - \mu_n) \underset{\sim}{b}_n .$$

But B is LI, so

$$\lambda_1 - \mu_1 = \dots = \lambda_n - \mu_n = 0 .$$

Hence

$$\lambda_1 = \mu_1 , \dots , \lambda_n = \mu_n ,$$

verifying uniqueness. ✓ $\square$

__General fact__ : Let $B = \{ \underset{\sim}{b}_1, \dots, \underset{\sim}{b}_n \}$ be a basis for a vector space V. Then

$$(\forall \underset{\sim}{v} \in V)\ (\exists\ \underline{\underline{\text{unique}}}\ \lambda_1, \dots, \lambda_n)$$

$$\underset{\sim}{v} = \lambda_1 \underset{\sim}{b}_1 + \dots + \lambda_n \underset{\sim}{b}_n .$$

We put

$$[\underset{\sim}{v}]_B = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

called the __coordinate vector__ (or __coordinates__) of $\underset{\sim}{v}$ with respect to B.

In particular, if $B = \{ \underset{\sim}{e}_1, \dots, \underset{\sim}{e}_n \}$ is the standard basis for F^n then

$$\left[(\lambda_1, \dots, \lambda_n) \right]_B = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} .$$

Finding the coordinate vector with respect to a basis often amounts to solving a system of equations.

Example: Verify that $B = \{(1,1), (2,-1)\}$ is a basis for $\mathbb{R}^2$ and find

$$[\underset{\sim}{v}]_B \quad \text{where} \quad \underset{\sim}{v} = (3,-7).$$

Solution: Certainly B is LI ✓, since neither $(1,1)$ nor $(2,-1)$ is a scalar multiple of the other.

To see that $\langle B \rangle = \mathbb{R}^2$, let $(x,y) \in \mathbb{R}^2$ and consider the equation

$$(x,y) = \alpha(1,1) + \beta(2,-1) = (\alpha + 2\beta, \alpha - \beta)$$

for some $\alpha, \beta \in \mathbb{R}$.

This is equivalent to the system

$$\boxed{\begin{aligned} \alpha + 2\beta &= x \\ \alpha - \beta &= y \end{aligned}} \qquad (\alpha, \beta \in \mathbb{R}).$$

<u>Solving</u>:

$$\left[\begin{array}{cc|c} 1 & 2 & x \\ 1 & -1 & y \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & -3 & y-x \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 1 & \frac{x-y}{3} \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & \frac{x+2y}{3} \\ 0 & 1 & \frac{x-y}{3} \end{array}\right]$$

This is equivalent to the system

$$\boxed{\begin{aligned} \alpha + 2\beta &= x \\ \alpha - \beta &= y \end{aligned}} \qquad (\alpha, \beta \in \mathbb{R}).$$

Solving:
$$\left[\begin{array}{cc|c} 1 & 2 & x \\ 1 & -1 & y \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{x+2y}{3} \\ 0 & 1 & \frac{x-y}{3} \end{array}\right]$$

so that $\alpha = \frac{x+2y}{3}$, $\beta = \frac{x-y}{3}$.

Since solutions for α, β always exist, for all $x, y \in \mathbb{R}$, then $\langle B \rangle = \mathbb{R}^2$, completing the verification that B is a basis for $\mathbb{R}^2$. ✓

Let $\underset{\sim}{v} = (3,-7) = (x,y)$ where $x=3$, $y=-7$.

Then, from above,

$$\alpha = \frac{x+2y}{3} = \frac{3-14}{3} = -\frac{11}{3},$$

$$\beta = \frac{x-y}{3} = \frac{3+7}{3} = \frac{10}{3},$$

so

$$\underset{\sim}{v} = -\frac{11}{3}(1,1) + \frac{10}{3}(2,-1),$$

so

$$[\underset{\sim}{v}]_B = \begin{bmatrix} -11/3 \\ 10/3 \end{bmatrix}.$$

Dimension of a vector space :

The dimension of a vector space V is the size of any basis for V.

This is a sensible definition, because of the following (deep) result :

Theorem : If B_1 and B_2 are bases for V then $|B_1| = |B_2|$.

$|X|$ denotes the size of a set X

We denote the dimension of V by

$$\dim(V).$$

e.g. $\dim(\mathbb{R}^2) = 2$, $\dim(\mathbb{R}^3) = 3$,

$\dim(F^n) = n$ size of the standard basis,

$\dim(\mathbb{P}_n) = n+1$

since $\{1, x, x^2, \dots, x^n\}$ is a basis for $\mathbb{P}_n$

spanning property holds since every polynomial is a linear combination of $1, x, \dots, x^n$

LI holds since every polynomial is uniquely determined by its coefficients

Rank of a matrix :

Let M be a matrix over a field F.

The **row rank** of M is the dimension of the row space of M.

$\underbrace{\text{row space}}$

span of the rows of M

The **column rank** of M is the dimension of the column space of M.

$\underbrace{\text{column space}}$

span of the columns of M

Theorem (difficult):

row rank of M = column rank of M.

obtain either by row reducing M or M^T to echelon form and counting the number of nonzero rows

Define the rank of M to be this common dimension:

$$\boxed{\text{rank } M = \text{row rank of } M = \text{column rank of } M}$$

Fact: The nonzero rows of any row echelon form of M form a basis for the row space of M.

- by manipulating M^T (transpose of M) and converting back to columns we can get a basis for the column space of M.

Example: Working over $\mathbb{Z}_2$, find bases for the row and column spaces of

$$M = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Example: Working over $\mathbb{Z}_2$, find bases for the row and column spaces of

$$M = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Solution:

$$M \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $\{ [1\ 0\ 1\ 1], [0\ 1\ 0\ 1], [0\ 0\ 1\ 0] \}$ is a basis for the row space of M.

$$\text{rank } M = 3$$

Also

$$M^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \right\}$

is a basis for the row space of M^T,

so $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

is a basis for the column space of M.

Example : Working over $\mathbb{R}$, find the rank of

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}.$$

Solution :

$$M \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the rank is 2. ✓

Check:

$$M^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 9 \\ 0 & -4 & -8 \\ 0 & -8 & -16 \\ 0 & -12 & -24 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with 2 nonzero rows. ✓