

MATH 2022 Linear and Abstract Algebra

LECTURE 26 Wednesday 01/05/2019

## Basis and dimension

- a basis for a vector space provides a foundation for accessing all vectors
- bases (plural of basis) of vector spaces facilitate matrix arithmetic
- choosing the right bases or modifying bases lead to simplified calculations & solutions
- dimension is the size of a basis of a vector space and captures precisely notions of "degrees of freedom"

Let  $V$  be a vector space over a field  $F$ .

A subset  $B$  of  $V$  is called a basis for  $V$  if

(i)  $B$  is a spanning set for  $V$ , that is,

$V = \langle B \rangle = \{ \text{all possible linear combinations of elements of } B \}$

so every  $\underline{v} \in V$  can be expressed as a

linear combination  $\underline{v} = \lambda_1 \underline{b}_1 + \dots + \lambda_n \underline{b}_n$

for some  $\underline{b}_1, \dots, \underline{b}_n \in B$  and  $\lambda_1, \dots, \lambda_n \in F$

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i.e.

for all distinct  $b_1, \dots, b_n \in B$ , we have

$$(\forall \lambda_1, \dots, \lambda_n \in F) \quad \lambda_1 b_1 + \dots + \lambda_n b_n = \underline{0} \Rightarrow \lambda_1 = \dots = \lambda_n = 0.$$

Note: this definition allows  $B$  to be infinite, though in most examples & applications,  $B$  will be finite.

Two ingredients of a basis:

(i) the spanning property guarantees that one can reach everything in  $V$  by applying vector space operations to elements of  $B$ .

(ii) LI of  $B$  guarantees that no element of  $B$  is superfluous.

Examples (standard bases):


$\{(1,0), (0,1)\}$  is a basis for  $F^2$ ,

$\{(1,0,0), (0,1,0), (0,0,1)\}$  " " " "  $F^3$ ,

More generally,  $\{\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n\}$  is a basis for  $F^n$

where

$$\underline{e}_i = (0, \dots, 0, 1, 0, \dots, 0) \quad \text{for } i=1, \dots, n,$$

  
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Note, if  $\underline{v} = (\lambda_1, \lambda_2, \dots, \lambda_n) \in F^n$  then

$$\underline{v} = \lambda_1 \underline{e}_1 + \lambda_2 \underline{e}_2 + \dots + \lambda_n \underline{e}_n,$$

so that the coordinates of  $\underline{v}$  become coefficients of the basis elements.

General fact : Let  $B = \{b_1, \dots, b_n\}$  be a basis for a vector space  $V$ . Then

$$(\forall \vec{v} \in V) (\exists \text{ unique } \lambda_1, \dots, \lambda_n)$$

$$\vec{v} = \lambda_1 b_1 + \dots + \lambda_n b_n.$$

Proof : That such linear combinations exist follows from the spanning property of  $B$ .

To prove uniqueness, suppose

$$\vec{v} = \lambda_1 b_1 + \dots + \lambda_n b_n = \mu_1 b_1 + \dots + \mu_n b_n$$

for some  $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n \in F$ .

Then

$$\begin{aligned} \underline{z} &= \underline{z} - \underline{z} \\ &= (\lambda_1 \underline{b}_1 + \dots + \lambda_n \underline{b}_n) - (\mu_1 \underline{b}_1 + \dots + \mu_n \underline{b}_n) \\ &= (\lambda_1 - \mu_1) \underline{b}_1 + \dots + (\lambda_n - \mu_n) \underline{b}_n . \end{aligned}$$

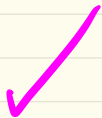
But  $\underline{B}$  is LI, so

$$\lambda_1 - \mu_1 = \dots = \lambda_n - \mu_n = 0 .$$

Hence

$$\lambda_1 = \mu_1 , \dots , \lambda_n = \mu_n ,$$

verifying uniqueness.



General fact : Let  $B = \{ \underline{b}_1, \dots, \underline{b}_n \}$  be a basis for a vector space  $V$ . Then

$$(\forall \underline{v} \in V) (\exists \text{ unique } \lambda_1, \dots, \lambda_n)$$

$$\underline{v} = \lambda_1 \underline{b}_1 + \dots + \lambda_n \underline{b}_n .$$

We put

$$[\underline{v}]_B = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

called the coordinate vector (or coordinates) of  $\underline{v}$  with respect to  $B$ .

In particular, if  $B = \{e_1, \dots, e_n\}$  is the standard basis for  $F^n$  then

$$[(\lambda_1, \dots, \lambda_n)]_B = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

Finding the coordinate vector with respect to a basis often amounts to solving a system of equations.



Example: Verify that  $B = \{(1,1), (2,-1)\}$  is a basis for  $\mathbb{R}^2$  and find

$$\begin{bmatrix} \underline{v} \\ \underline{w} \end{bmatrix}_B \quad \text{where} \quad \underline{v} = (3, -7).$$

Solution: Certainly  $B$  is LI, since neither  $(1,1)$  nor  $(2,-1)$  is a scalar multiple of the other.

To see that  $\langle B \rangle = \mathbb{R}^2$ , let  $(x,y) \in \mathbb{R}^2$  and consider the equation

$$(x,y) = \alpha(1,1) + \beta(2,-1) = (\alpha + 2\beta, \alpha - \beta)$$

for some  $\alpha, \beta \in \mathbb{R}$ .

This is equivalent to the system

$$\begin{cases} \alpha + 2\beta = x \\ \alpha - \beta = y \end{cases}$$

$$(\alpha, \beta \in \mathbb{R})$$

Solving:

$$\left[ \begin{array}{cc|c} 1 & 2 & x \\ 1 & -1 & y \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 2 & x \\ 0 & -3 & y-x \end{array} \right]$$

$$\sim \left[ \begin{array}{cc|c} 1 & 2 & x \\ 0 & 1 & \frac{x-y}{3} \end{array} \right]$$

$$\sim \left[ \begin{array}{cc|c} 1 & 0 & \frac{x+2y}{3} \\ 0 & 1 & \frac{x-y}{3} \end{array} \right]$$

This is equivalent to the system

$$\begin{cases} \alpha + 2\beta = x \\ \alpha - \beta = y \end{cases}$$

$$(\alpha, \beta \in \mathbb{R})$$

Solving:

$$\left[ \begin{array}{cc|c} 1 & 2 & x \\ 1 & -1 & y \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 0 & \frac{x+2y}{3} \\ 0 & 1 & \frac{x-y}{3} \end{array} \right]$$

so that  $\alpha = \frac{x+2y}{3}$ ,  $\beta = \frac{x-y}{3}$ .

Since solutions for  $\alpha, \beta$  always exist, for all  $x, y \in \mathbb{R}$ , then  $\langle B \rangle = \mathbb{R}^2$ , completing the verification that  $B$  is a basis for  $\mathbb{R}^2$ . ✓

Let  $\underline{v} = (3, -7) = (x, y)$  where  $x=3$ ,  $y=-7$ .

Then, from above,

$$\alpha = \frac{x+2y}{3} = \frac{3-14}{3} = -\frac{11}{3},$$

$$\beta = \frac{x-y}{3} = \frac{3+7}{3} = \frac{10}{3},$$

so

$$\underline{v} = -\frac{11}{3} (1, 1) + \frac{10}{3} (2, -1),$$

so

$$[\underline{v}]_B = \begin{bmatrix} -11/3 \\ 10/3 \end{bmatrix}.$$

## Dimension of a vector space :

The dimension of a vector space  $V$  is the size of any basis for  $V$ .

This is a sensible definition, because of the following (deep) result :

Theorem : If  $B_1$  and  $B_2$  are bases for  $V$   
then  $|B_1| = |B_2|$ .

$|X|$  denotes the size of a set  $X$

We denote the dimension of  $V$  by  
 $\dim(V)$ .

e.g.  $\dim(\mathbb{R}^2) = 2$ ,  $\dim(\mathbb{R}^3) = 3$ ,

$$\dim(F^n) = n$$

size of the standard basis

$$\dim(P_n) = n+1$$

since  $\{1, x, x^2, \dots, x^n\}$  is a basis for  $P_n$

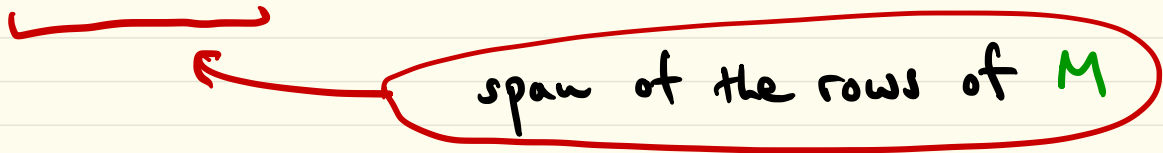
spanning property holds since every polynomial is a linear combination of  $1, x, \dots, x^n$

LI holds since every polynomial is uniquely determined by its coefficients

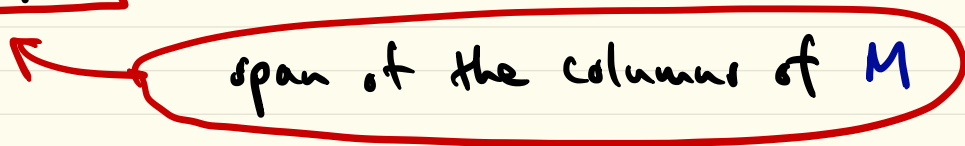
## Rank of a matrix :

Let  $M$  be a matrix over a field  $F$ .

The row rank of  $M$  is the dimension of the row space of  $M$ .

span of the rows of  $M$

The column rank of  $M$  is the dimension of the column space of  $M$ .

span of the columns of  $M$

Theorem (difficult) :

row rank of  $M$  = column rank of  $M$ .

obtain either by row reducing  $M$  or  $M^T$   
to echelon form and counting the  
number of nonzero rows

Define the rank of  $M$  to be this common dimension:

$\text{rank } M = \text{row rank of } M = \text{column rank of } M$



Fact: The nonzero rows of any row echelon form of  $M$  form a basis for the row space of  $M$ .

— by manipulating  $M^T$  (transpose of  $M$ ) and converting back to columns we can get a basis for the column space of  $M$ .

Example: Working over  $\mathbb{Z}_2$ , find bases for the row and column spaces of

$$M = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Example: Working over  $\mathbb{Z}_2$ , find bases for the row and column spaces of

$$M = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Solution:

$$M \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{so } \{ [1 \ 0 \ 1 \ 1], [0 \ 1 \ 0 \ 1], [0 \ 0 \ 1 \ 0] \}$$

is a basis for the row space of  $M$ .

$$\text{rank } M = 3$$

Also

$$M^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so  $\{ [1 \ 0 \ 1 \ 0], [0 \ 1 \ 0 \ 1], [0 \ 0 \ 1 \ 1] \}$

is a basis for the row space of  $M^T$ ,

so  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

is a basis for the column space of  $M$ .

Example : Working over  $\mathbb{R}$ , find the rank of

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}.$$

Solution :  $M \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

so the rank is  $2$  . ✓

Check :  $M^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 9 \\ 0 & -4 & -8 \\ 0 & -8 & -16 \\ 0 & -12 & -24 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

with  $2$  nonzero rows. ✓