MATH 2022 Linear and Abstract Algebra

LECTURE 30 Thursday 09/05/2019

$\boxed{\begin{array}{l}\underline{\text{Key property of matrix exponentials}}: \\ \quad M = PDP^{-1} \implies e^M = P e^D P^{-1}\end{array}}$

Proof: Suppose $M = PDP^{-1}$.

Note first that, for any positive integer k,

$$M^k = \underbrace{(PDP^{-1})(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}_{k \text{ instances}}$$

$$= P\underbrace{DDD\cdots D}_{k \text{ instances}}P^{-1} = PD^kP^{-1}.$$

Then

$$e^M = I + M + \frac{1}{2!} M^2 + \frac{1}{3!} M^3 + \dots + \frac{1}{n!} M^n + \dots$$

$$= I + PDP^{-1} + \frac{1}{2!} (PDP^{-1})^2 + \frac{1}{3!} (PDP^{-1})^3 +$$

$$\dots + \frac{1}{n!} (PDP^{-1})^n + \dots$$

$$= PIP^{-1} + PDP^{-1} + P\left(\frac{1}{2!} D^2\right)P^{-1} + P\left(\frac{1}{3!} D^3\right)P^{-1} +$$

$$\dots + P\left(\frac{1}{n!} D^n\right)P^{-1} + \dots$$

$$= P\left(I + D + \frac{1}{2!} D^2 + \frac{1}{3!} D^3 + \dots + \frac{1}{n!} D^n + \dots\right) P^{-1}$$

$$= P e^D P^{-1} .$$

$\square$

Linear transformations & matrices:

Let $T: V \to W$ be a linear transformation where

$$\begin{cases} \mathcal{B} = \{\underset{\sim}{b}_1, \dots, \underset{\sim}{b}_n\} \text{ is a basis for } V \\ \mathcal{D} = \{\underset{\sim}{d}_1, \dots, \underset{\sim}{d}_m\} \text{ " " " " } W. \end{cases}$$

Recall that the matrix of T with respect to $\mathcal{B}$ and $\mathcal{D}$ is

$$[T]_{\mathcal{D}}^{\mathcal{B}} = \left[[T(\underset{\sim}{b}_1)]_{\mathcal{D}} \; \cdots \; [T(\underset{\sim}{b}_n)]_{\mathcal{D}} \right].$$

columns of coordinates with respect to $\mathcal{D}$

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$$\boxed{\underline{\text{Theorem}}: \quad (\forall \underset{\sim}{v} \in V) \quad [T(\underset{\sim}{v})]_{\mathcal{D}} = [T]_{\mathcal{D}}^{\mathcal{B}} [\underset{\sim}{v}]_{\mathcal{B}}}$$

action of T on $\underset{\sim}{v}$

matrix multiplication

Composition & matrix multiplication:

Consider linear transformations

$$\begin{tikzcd} \mathcal{U} \arrow[r, "S"] \arrow[rr, bend right, "T \circ S"'] & V \arrow[r, "T"] & W \end{tikzcd}$$

with finite (ordered) bases A, B, C for $\mathcal{U}, V, W$ respectively.

$$\boxed{\underline{\textbf{Theorem}}: \quad [T \circ S]_C^A = [T]_C^B \, [S]_B^A}$$

Proof: Let $A = \{\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_q\}$.

By definition, for $j = 1, \ldots, q$, the jth column of $[T \circ S]_C^A$ is the column of coordinates

$$\begin{aligned}
[(T \circ S)(\underset{\sim}{a}_j)]_C &= [T(S(\underset{\sim}{a}_j))]_C \\
&= [T]_C^B \, [S(\underset{\sim}{a}_j)]_B \\
&= [T]_C^B \, [S]_B^A \, [\underset{\sim}{a}_j]_A \\
&= [T]_C^B \, [S]_B^A \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\end{aligned}$$

jth place

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&= [T]^B_C \, [S(\underset{\sim}{a}_j)]_B \\
&= [T]^B_C \, [S]^A_B \, [\underset{\sim}{a}_j]_A \\
&= [T]^B_C \, [S]^A_B \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{jth place}
\end{aligned}
$$

which is the jth column of the matrix $[T]^B_C [S]^A_B$.

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Since the jth columns match for $j = 1, \ldots, q$, we have

$$[T \circ S]^A_C = [T]^B_C [S]^A_B .$$

$\checkmark$ $\square$

Change of basis matrices:

Let V be a vector space with bases B, D.

Recall the identity mapping

$$\mathrm{id} : V \to V, \quad \underset{\sim}{v} \mapsto \underset{\sim}{v} \quad (\forall \underset{\sim}{v} \in V).$$

We saw that

$$[\mathrm{id}]^B_B = [\mathrm{id}]^D_D = I$$

$\boxed{\text{What about } [\mathrm{id}]^B_D \text{ and } [\mathrm{id}]^D_B \text{ ?}}$

change of basis matrices

Example: Let

$$B = \{(1,0), (0,1)\}, \quad D = \{(1,1), (1,-1)\},$$

both bases for $\mathbb{R}^2$. Then

$$[id]_B^D = \Big[[id(1,1)]_B \quad [id(1,-1)]_B \Big]$$

$$= \Big[[(1,1)]_B \quad [(1,-1)]_B \Big]$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

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$$[\mathrm{id}]_B^D = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Observe that

$$(1,0) = \tfrac{1}{2}(1,1) + \tfrac{1}{2}(1,-1),$$

$$(0,1) = \tfrac{1}{2}(1,1) - \tfrac{1}{2}(1,-1),$$

so that

$$[(1,0)]_D = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad [(0,1)]_D = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}.$$

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Hence

$$[id]_D^B = \Big[\, [id(1,0)]_D \;\; [id(0,1)]_D \,\Big]$$
$$= \Big[\, [(1,0)]_D \;\; [(0,1)]_D \,\Big]$$
$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1/2 \end{bmatrix}$$

mutually inverse matrices

no accident!!

Theorem: Let B, D be bases for V.

Then

(i) $[\mathrm{id}]_D^B \, [\underset{\sim}{v}]_B = [\underset{\sim}{v}]_D \quad \forall \underset{\sim}{v} \in V$

(ii) $[\mathrm{id}]_D^B$ and $[\mathrm{id}]_B^D$ are mutually inverse.

facilitates finding new coordinates after changing the basis

interchanging bases inverts the matrix

__Proof__ : (i) By an earlier theorem,

$$[\mathrm{id}]^{B}_{D}\,[\underset{\sim}{v}]_{B} = [\,\mathrm{id}(\underset{\sim}{v})\,]_{D} = [\underset{\sim}{v}]_{D}\,. \checkmark$$

(ii) By another earlier theorem,

$$[\mathrm{id}]^{D}_{B}\,[\mathrm{id}]^{B}_{D} = [\,\mathrm{id}\circ\mathrm{id}\,]^{B}_{B} = [\mathrm{id}]^{B}_{B} = I_{n}$$

and similarly $[\mathrm{id}]^{B}_{D}\,[\mathrm{id}]^{D}_{B} = [\mathrm{id}]^{D}_{D} = I_{n}$,

so that $[\mathrm{id}]^{B}_{D}$ and $[\mathrm{id}]^{D}_{B}$ are mutually inverse.

$\checkmark$ $\square$

Application to diagonalisation

Let $T : V \to V$ be a linear operator with bases B and D for V. Then

$$[T]_B^B = [\mathrm{id} \circ T \circ \mathrm{id}]_B^B$$

$$= [\mathrm{id}]_B^D \, [T \circ \mathrm{id}]_D^B$$

$$= [\mathrm{id}]_B^D \, [T]_D^D \, [\mathrm{id}]_D^B$$

so

$$\boxed{[T]_B^B = P \, [T]_D^D \, P^{-1}}$$

if we put $P = [\mathrm{id}]_B^D$, so that $P^{-1} = [\mathrm{id}]_D^B$.

$$\boxed{[T]^B_B = P\,[T]^D_D\,P^{-1}}$$

typical to use standard basis B for F^w

typical to look for basis D such that $[T]^D_D$ is <u>diagonal</u>

then the columns of P become eigenvectors for the matrix $[T]^B_B$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$T(x,y) = (5x-4y,\ 3x-2y).$$

Find a basis D for which $[T]_D^D$ is diagonal, and then find a formula for

$$T^n(x,y) \quad \text{for any } n \geq 1.$$

Solution: Let $B = \{(1,0), (0,1)\}$, the standard basis for $\mathbb{R}^2$. Then

$$[T]_B^B = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}.$$

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Easy to check eigenvalues are 1, 2 with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ respectively.

Let $D = \{ (1,1), (4,3) \}$, another basis for $\mathbb{R}^2$, and put

$$P = [id]_B^D = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}.$$

Then

$$[id]_D^B = P^{-1} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix}$$

Recall, $T(x,y) = (5x-4y,\ 3x-2y)$,

so $T(1,1) = (1,1)$ and $T(4,3) = (8,6)$,

giving
$$[T]_D^D = \Big[\, [T(1,1)]_D \quad [T(4,3)]_D \,\Big]$$
$$= \Big[\, [(1,1)]_D \quad [(8,6)]_D \,\Big]$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

diagonal

We want a formula for $T^n(x,y)$, but first look at
$$[T^n(x,y)]_D.$$

coordinates with respect to D

We have

$$
\begin{aligned}
[T^n(x,y)]_D &= [T^n]_D^D \, [(x,y)]_D \\
&= \left([T]_D^D\right)^n [id]_D^B \, [(x,y)]_B \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^n \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} -3x+4y \\ x-y \end{bmatrix} \\
&= \begin{bmatrix} -3x+4y \\ 2^n(x-y) \end{bmatrix}.
\end{aligned}
$$

We have

$$[T^n(x,y)]_D = \begin{bmatrix} -3x+4y \\ 2^n(x-y) \end{bmatrix}.$$

Hence

$$T^n(x,y) = (-3x+4y)(1,1) + 2^n(x-y)(4,3)$$

$$= (-3x+4y+4(2^n)(x-y), -3x+4y+3(2^n)(x-y))$$

$$= \left((-3+2^{n+2})x + 4(1-2^n)y , 3(2^n-1)x + (4-3(2^n))y \right).$$

not something one could ever guess !!