MATH 2022 Linear and Abstract Algebra

LECTURE 32 Wednesday 15/05/2019

Real inner product spaces

- abstracts familiar properties of dot products
- listed as axioms for an inner product space
- applications to important spaces of functions, using theory of integration
- general notions of length & distance
- leading to the celebrated Cauchy-Schwarz inequality
- notions of angles & orthogonality
- general explanation of the triangle inequality

Let V be a vector space over the field $\mathbb{R}$.

An inner product on V is a mapping

$$\langle \;,\; \rangle : \underbrace{V \times V}_{\text{Cartesian product } \{(\underset{\sim}{v}, \underset{\sim}{w}) \mid \underset{\sim}{v}, \underset{\sim}{w} \in V\}} \longrightarrow \mathbb{R}$$

gaps for inputs from V

such that the following five axioms hold:

(1)

(2)

(3)

(4)

(5)

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We then call V an inner product space.

such that the following five axioms hold:

(1) $(\forall \underset{\sim}{v}, \underset{\sim}{w} \in V) \quad \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = \langle \underset{\sim}{w}, \underset{\sim}{v} \rangle$ (commutativity)

(2) $(\forall \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V) \quad \langle \underset{\sim}{u} + \underset{\sim}{v}, \underset{\sim}{w} \rangle = \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle + \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$ (distributivity)

(3) $(\forall \underset{\sim}{v}, \underset{\sim}{w} \in V)(\forall \lambda \in \mathbb{R}) \quad \langle \lambda \underset{\sim}{v}, \underset{\sim}{w} \rangle = \lambda \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$

(4) $(\forall \underset{\sim}{v} \in V) \quad \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle \geqslant 0$

(5) $(\forall \underset{\sim}{v} \in V) \quad \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = 0 \iff \underset{\sim}{v} = \underset{\sim}{0}$.

Examples :

(i) $\mathbb{R}^n$ is an inner product space where

$$\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = \underset{\sim}{v} \cdot \underset{\sim}{w}$$

usual dot product

where

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n .$$

(ii) Let $V = \{ \text{continuous functions} : [a,b] \to \mathbb{R} \}$ where $a < b$ fixed. (V is a subspace of $\mathbb{R}^{[a,b]}$.)

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Then V is a vector space with respect to addition & scalar multiplication of functions. Define

$$\langle \, , \, \rangle : V \times V \longrightarrow \mathbb{R}$$

by, for $f, g \in V$,

$$\boxed{\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx .}$$

"continuous" dot product

It is routine to verify the inner product axioms (1)–(4):

For example,

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$$

$$= \int_a^b g(x)\, f(x)\, dx = \langle g, f \rangle,$$

so (1) holds, and

$$\langle f, f \rangle = \int_a^b f(x)\, f(x)\, dx$$

$$= \int_a^b \left(f(x)\right)^2 dx \geq 0,$$

since the integrand is nonnegative, so (4) holds.

Axiom (5) is trickier:

Suppose $\langle f, f \rangle = \int_a^b (f(x))^2 \, dx = 0$.

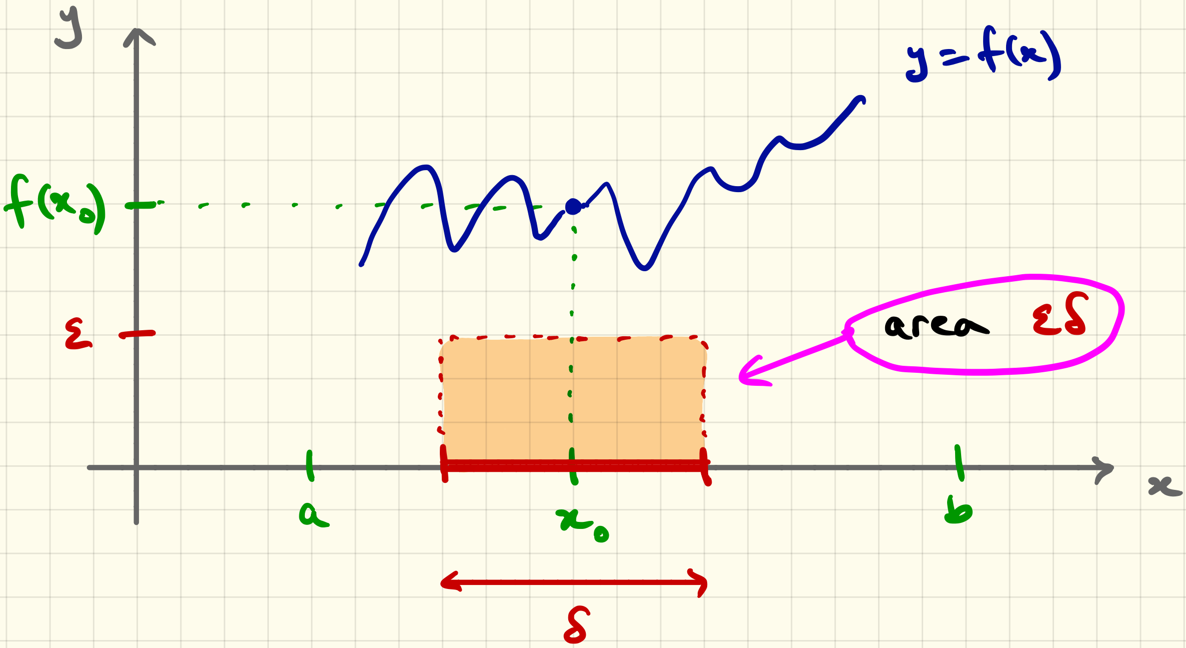
We want to show $f = \underset{\sim}{0}$, the zero function.

Suppose, to the contrary, that $f \neq \underset{\sim}{0}$,

so $f(x_0) \neq 0$ for some $x_0 \in [a,b]$.

Continuity of f guarantees that there exists some $\varepsilon > 0$ and some interval I of positive width $\delta > 0$ such that

$$|f(x)| \geqslant \varepsilon \quad \text{for all } x \in I.$$



Then $\int_a^b (f(x))^2\,dx \geq \varepsilon^2\delta > 0$,

contradicting that $\langle f,f\rangle = 0$.

Hence $f = \underset{\sim}{0}$, and axiom (5) holds. ✓

Some important elementary consequences of the axioms:

Let V be an inner product space. Then

(a) $(\forall \underset{\sim}{v} \in V) \quad \langle \underset{\sim}{0}, \underset{\sim}{v} \rangle = \langle \underset{\sim}{v}, \underset{\sim}{0} \rangle = 0$

zero vector

real number

(b) $(\forall \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V) \quad \langle \underset{\sim}{u}, \underset{\sim}{v} + \underset{\sim}{w} \rangle = \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle + \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle$

(c) $(\forall \underset{\sim}{v}, \underset{\sim}{w} \in V)(\forall \lambda \in \mathbb{R}) \quad \langle \underset{\sim}{v}, \lambda \underset{\sim}{w} \rangle = \lambda \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$.

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<u>Proofs</u>: (a) Let $\underset{\sim}{v} \in V$. By axiom (2),

$$\langle \underset{\sim}{0}, \underset{\sim}{v} \rangle = \langle \underset{\sim}{0} + \underset{\sim}{0}, \underset{\sim}{v} \rangle = \langle \underset{\sim}{0}, \underset{\sim}{v} \rangle + \langle \underset{\sim}{0}, \underset{\sim}{v} \rangle,$$

so $\langle \underset{\sim}{0}, \underset{\sim}{v} \rangle = 0$ (only real number with this property).

Also $\langle \underset{\sim}{v}, \underset{\sim}{0} \rangle = \langle \underset{\sim}{0}, \underset{\sim}{v} \rangle = 0$, using axiom (1). ✓

(b) Let $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V$. Then

$$\langle \underset{\sim}{u}, \underset{\sim}{v}+\underset{\sim}{w} \rangle = \langle \underset{\sim}{v}+\underset{\sim}{w}, \underset{\sim}{u} \rangle \quad \text{by axiom (1)}$$

$$= \langle \underset{\sim}{v}, \underset{\sim}{u} \rangle + \langle \underset{\sim}{w}, \underset{\sim}{u} \rangle \quad \text{by axiom (2)}$$

$$= \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle + \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle \quad \text{by axiom (1).}$$

✓

(c) Let $\underset{\sim}{v}, \underset{\sim}{w} \in V$ and $\lambda \in \mathbb{R}$. Then

$$\langle \underset{\sim}{v}, \lambda \underset{\sim}{w} \rangle = \langle \lambda \underset{\sim}{w}, \underset{\sim}{v} \rangle \quad \text{by axiom (1)}$$

$$= \lambda \langle \underset{\sim}{w}, \underset{\sim}{v} \rangle \quad \text{by axiom (3)}$$

$$= \lambda \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle \quad \text{by axiom (1).}$$

✓ $\square$

Length and the Cauchy-Schwarz inequality:

Let $\langle\, , \rangle$ be an inner product on a vector space V.

Define the **length** or **norm** of $\underset{\sim}{v} \in V$ by

$$\boxed{\|\underset{\sim}{v}\| \overset{\text{def}}{=} \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle^{1/2}}.$$

This generalises the definition in $\mathbb{R}^n$:

$$\boxed{\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2}}.$$

Recall, for geometric vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$,

$$\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = \underset{\sim}{v} \cdot \underset{\sim}{w} = \|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\| \cos\theta$$

where θ is the angle between $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

Then

$$\begin{aligned} |\underset{\sim}{v} \cdot \underset{\sim}{w}| &= \big| \|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\| \cos\theta \big| \\ &= \|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\| \, |\cos\theta| \\ &\le \|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\| \end{aligned}$$

since $|\cos\theta| \le 1$

called the Cauchy-Schwarz inequality.

It holds in complete generality:

Cauchy-Schwarz Inequality:

Let $\underset{\sim}{v}, \underset{\sim}{w} \in V$ where V is an inner product space.

Then

$$|\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle| \leq \|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\| .$$

Proof: If $\underset{\sim}{w} = \underset{\sim}{0}$ then

$$|\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle| = |\langle \underset{\sim}{v}, \underset{\sim}{0} \rangle| = |0| = 0 \leq \|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\|$$

trivially. ✓

Suppose then that $\underset{\sim}{w} \neq \underset{\sim}{0}$, so $\langle \underset{\sim}{w}, \underset{\sim}{w} \rangle \neq 0$.

Put

$$\lambda = \frac{\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle}{\| \underset{\sim}{w} \|^2} \in \mathbb{R} .$$

Then

$$0 \leq \langle \underset{\sim}{v} - \lambda \underset{\sim}{w}, \underset{\sim}{v} - \lambda \underset{\sim}{w} \rangle$$

by axiom (4)

$$= \langle \underset{\sim}{v}, \underset{\sim}{v} - \lambda \underset{\sim}{w} \rangle - \lambda \langle \underset{\sim}{w}, \underset{\sim}{v} - \lambda \underset{\sim}{w} \rangle$$

by axioms (2), (3)

$$= \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle - \lambda \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle - \lambda \langle \underset{\sim}{w}, \underset{\sim}{v} \rangle + \lambda^2 \langle \underset{\sim}{w}, \underset{\sim}{w} \rangle$$

by properties (b), (c)

$$= \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle - 2\lambda \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle + \lambda^2 \langle \underset{\sim}{w}, \underset{\sim}{w} \rangle$$

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Put
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Then
$$0 \le \langle \underset{\sim}{v} - \lambda \underset{\sim}{w}, \underset{\sim}{v} - \lambda \underset{\sim}{w} \rangle$$
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$$= \langle \underset{\sim}{v}, \underset{\sim}{v} - \lambda \underset{\sim}{w} \rangle - \lambda \langle \underset{\sim}{w}, \underset{\sim}{v} - \lambda \underset{\sim}{w} \rangle$$
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by axiom (1)

$$= \| \underset{\sim}{v} \|^2 - 2 \frac{\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle^2}{\| \underset{\sim}{w} \|^2} + \frac{\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle^2}{\| \underset{\sim}{w} \|^4} \| \underset{\sim}{w} \|^2$$

$$= \| \underset{\sim}{v} \|^2 - \frac{\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle^2}{\| \underset{\sim}{w} \|^2} .$$

Hence
$$\|\underset{\sim}{v}\|^2 - \frac{\langle \underset{\sim}{v}, \underset{\sim}{w}\rangle^2}{\|\underset{\sim}{w}\|^2} \geqslant 0,$$
so
$$\|\underset{\sim}{v}\|^2 \geqslant \frac{\langle \underset{\sim}{v}, \underset{\sim}{w}\rangle^2}{\|\underset{\sim}{w}\|^2},$$
so
$$\|\underset{\sim}{v}\|^2 \|\underset{\sim}{w}\|^2 \geqslant \langle \underset{\sim}{v}, \underset{\sim}{w}\rangle^2,$$
i.e. $$\langle \underset{\sim}{v}, \underset{\sim}{w}\rangle^2 \leq \|\underset{\sim}{v}\|^2 \|\underset{\sim}{w}\|^2,$$
so, taking nonnegative square roots,
$$|\langle \underset{\sim}{v}, \underset{\sim}{w}\rangle| \leq \|\underset{\sim}{v}\| \|\underset{\sim}{w}\|,$$
and we are done. ✓ $\square$

It holds in complete generality:

Cauchy-Schwarz inequality:

Let $\underset{\sim}{v}, \underset{\sim}{w} \in V$ where V is an inner product space.

Then

$$\left| \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle \right| \leq \| \underset{\sim}{v} \| \, \| \underset{\sim}{w} \| .$$

Special case: If $\underset{\sim}{v} = (v_1, \dots, v_n)$, $\underset{\sim}{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$

then the Cauchy-Schwarz inequality gives

$$\boxed{\left| v_1 w_1 + \dots + v_n w_n \right| \leq \left(v_1^2 + \dots + v_n^2 \right)^{1/2} \left(w_1^2 + \dots + w_n^2 \right)^{1/2}} .$$

Angles and orthogonality:

Rearranging Cauchy-Schwarz for $\underset{\sim}{v}, \underset{\sim}{w} \neq \underset{\sim}{0}$ yields

$$\frac{|\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle|}{\|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\|} \leq 1,$$

so

$$-1 \leq \frac{\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle}{\|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\|} \leq 1,$$

so there is a unique $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle}{\|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\|}.$$

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$$\cos\theta = \frac{\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle}{\|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\|}.$$

> We call this θ the <u>angle between</u> $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

If $\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = 0$ then $\theta = \pi/2$ and we say that $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are orthogonal or mutually perpendicular.

A set of vectors in an inner product space is called orthogonal if every pair of distinct vectors in the set are orthogonal.

A set of vectors is called orthonormal if the set is orthogonal and every vector in the set has length 1.

Orthonormal bases play a critical role in applications of inner product spaces.

<u>Example</u>: Let $V = \{\text{continuous functions}: [-\pi,\pi] \to \mathbb{R}\}$ with inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

For $n \geq 1$, put

$$f_n(x) = \sin(nx), \quad g_n(x) = \cos(nx)$$

and

$$g_0(x) = 1 \quad (= \cos(0x)).$$

Then

$$\{g_0, g_1, g_2, \ldots, g_n, \ldots, f_1, f_2, \ldots, f_n, \ldots\}$$

is an orthogonal set of vectors.

Proof is an exercise in trig identities & integration.