MATH 2022 Linear and Abstract Algebra

LECTURE 36 Thursday 23/05/2019

The Gram-Schmidt process

- take any (finite) basis for a subspace W of an inner product space V
- carefully & methodically replace vectors in the basis, building up a sequence of vectors, such that
- vectors in the sequence are pairwise orthogonal
- " " " " have length 1
- extend the sequence one vector at a time, using a recursive formula and properties of projections

The Gram-Schmidt process

- take any (finite) basis for a subspace W of an inner product space V
- carefully & methodically replace vectors in the basis, building up a sequence of vectors, such that
- vectors in the sequence are pairwise orthogonal
- " " " " have length 1
- extend the sequence one vector at a time, using a recursive formula and properties of projections

producing finally an orthonormal basis for W.

The Gram-Schmidt process:

Let $B = \{ \underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n \}$ be a basis for an inner product space V.

We construct an orthonormal basis for V as follows:

Step (1): Normalise $\underset{\sim}{v}_1$ by putting

$$\underset{\sim}{b}_1 = \hat{\underset{\sim}{v}}_1 = \frac{\underset{\sim}{v}_1}{\|\underset{\sim}{v}_1\|}$$

so that $\|\underset{\sim}{b}_1\| = 1$ and $\{\underset{\sim}{b}_1\}$ is an orthonormal basis for

$$W_1 = \text{span of } \{\underset{\sim}{b}_1\} = \text{span of } \{\underset{\sim}{v}_1\}.$$

one-dimensional

Step (2): Form

$$\underset{\sim}{w}_2 = \underset{\sim}{v}_2 - \text{proj}_{W_1} \underset{\sim}{v}_2 = \underset{\sim}{v}_2 - \langle \underset{\sim}{v}_2, \underset{\sim}{b}_1 \rangle \underset{\sim}{b}_1 ,$$

which the general theory guarantees is orthogonal to W_1.
Then normalise $\underset{\sim}{w}_2$ by putting

$$\underset{\sim}{b}_2 = \hat{\underset{\sim}{w}}_2 = \frac{\underset{\sim}{w}_2}{\|\underset{\sim}{w}_2\|} = \frac{\underset{\sim}{v}_2 - \langle \underset{\sim}{v}_2, \underset{\sim}{b}_1 \rangle \underset{\sim}{b}_1}{\| \underset{\sim}{v}_2 - \langle \underset{\sim}{v}_2, \underset{\sim}{b}_1 \rangle \underset{\sim}{b}_1 \|} ,$$

so that $\|\underset{\sim}{b}_2\| = 1$ and $\{ \underset{\sim}{b}_1, \underset{\sim}{b}_2 \}$ is an orthonormal basis for

$$W_2 = \text{span of } \{ \underset{\sim}{b}_1, \underset{\sim}{b}_2 \} = \text{span of } \{ \underset{\sim}{v}_1, \underset{\sim}{v}_2 \} .$$

two-dimensional

Step (3): Form

$$\underset{\sim}{w}_3 = \underset{\sim}{v}_3 - \mathrm{proj}_{W_2} \underset{\sim}{v}_3 = \underset{\sim}{v}_3 - \langle \underset{\sim}{v}_3, \underset{\sim}{b}_1 \rangle \underset{\sim}{b}_1 - \langle \underset{\sim}{v}_3, \underset{\sim}{b}_2 \rangle \underset{\sim}{b}_2,$$

which the general theory guarantees is orthogonal to W_2.

Then normalise $\underset{\sim}{w}_3$ by putting

$$\underset{\sim}{b}_3 = \hat{\underset{\sim}{w}}_3 = \frac{\underset{\sim}{w}_3}{\|\underset{\sim}{w}_3\|} = \frac{\underset{\sim}{v}_3 - \langle \underset{\sim}{v}_3, \underset{\sim}{b}_1 \rangle \underset{\sim}{b}_1 - \langle \underset{\sim}{v}_3, \underset{\sim}{b}_2 \rangle \underset{\sim}{b}_2}{\| \underset{\sim}{v}_3 - \langle \underset{\sim}{v}_3, \underset{\sim}{b}_1 \rangle \underset{\sim}{b}_1 - \langle \underset{\sim}{v}_3, \underset{\sim}{b}_2 \rangle \underset{\sim}{b}_2 \|},$$

so that $\|\underset{\sim}{b}_3\| = 1$ and $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\}$ is an orthonormal basis for

$$W_3 = \text{span of } \{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\} = \text{span of } \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}.$$

three-dimensional

The process continues. The **recursive step** is

Step (k+1): Given an orthonormal basis $\{\underset{\sim}{b}_1, \dots, \underset{\sim}{b}_k\}$ for the subspace W_k spanned by $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k\}$, we create an orthonormal basis for the subspace W_{k+1} spanned by $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k, \underset{\sim}{v}_{k+1}\}$.

Put

$$\underset{\sim}{w}_{k+1} = \underset{\sim}{v}_{k+1} - \mathrm{proj}_{W_k} \underset{\sim}{v}_{k+1}$$

$$= \underset{\sim}{v}_{k+1} - \langle \underset{\sim}{v}_{k+1}, \underset{\sim}{b}_1 \rangle \underset{\sim}{b}_1 - \dots - \langle \underset{\sim}{v}_{k+1}, \underset{\sim}{b}_k \rangle \underset{\sim}{b}_k ,$$

which the theory guarantees is orthogonal to W_k.

Note that $\underset{\sim}{w}_{k+1} \neq \underset{\sim}{0}$, since $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k, \underset{\sim}{v}_{k+1}\}$ is LI.

The process continues. The **recursive step** is

Step (k+1): Given an orthonormal basis $\{\underset{\sim}{b}_1, \dots, \underset{\sim}{b}_k\}$ for the subspace W_k spanned by $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k\}$, we create an orthonormal basis for the subspace W_{k+1} spanned by $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k, \underset{\sim}{v}_{k+1}\}$.

Put

$$\begin{aligned} \underset{\sim}{w}_{k+1} &= \underset{\sim}{v}_{k+1} - \operatorname{proj}_{W_k} \underset{\sim}{v}_{k+1} \\ &= \underset{\sim}{v}_{k+1} - \langle \underset{\sim}{v}_{k+1}, \underset{\sim}{b}_1 \rangle \underset{\sim}{b}_1 - \cdots - \langle \underset{\sim}{v}_{k+1}, \underset{\sim}{b}_k \rangle \underset{\sim}{b}_k, \end{aligned}$$

which the theory guarantees is orthogonal to W_k.

Note that $\underset{\sim}{w}_{k+1} \neq \underset{\sim}{0}$, since $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k, \underset{\sim}{v}_{k+1}\}$ is LI.

Then normalise: $\underset{\sim}{b}_{k+1} = \hat{\underset{\sim}{w}}_{k+1} = \dfrac{\underset{\sim}{w}_{k+1}}{\|\underset{\sim}{w}_{k+1}\|}$.

Then $\{\underset{\sim}{b}_1, \dots, \underset{\sim}{b}_k, \underset{\sim}{b}_{k+1}\}$ is an orthonormal basis for W_{k+1}.

The process continues. The recursive step is

Step (k+1): Given an orthonormal basis $\{\underset{\sim}{b}_1, \dots, \underset{\sim}{b}_k\}$ for the subspace W_k spanned by $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k\}$, we create an orthonormal basis for the subspace W_{k+1} spanned by $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k, \underset{\sim}{v}_{k+1}\}$.

Put

$$\underset{\sim}{w}_{k+1} = \underset{\sim}{v}_{k+1} - \text{proj}_{W_k} \underset{\sim}{v}_{k+1}$$

$$= \underset{\sim}{v}_{k+1} - \langle \underset{\sim}{v}_{k+1}, \underset{\sim}{b}_1 \rangle \underset{\sim}{b}_1 - \cdots - \langle \underset{\sim}{v}_{k+1}, \underset{\sim}{b}_k \rangle \underset{\sim}{b}_k,$$

which the theory guarantees is orthogonal to W_k.

Note that $\underset{\sim}{w}_{k+1} \neq \underset{\sim}{0}$, since $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k, \underset{\sim}{v}_{k+1}\}$ is LI.

Then normalise: $\underset{\sim}{b}_{k+1} = \hat{\underset{\sim}{w}}_{k+1} = \dfrac{\underset{\sim}{w}_{k+1}}{\|\underset{\sim}{w}_{k+1}\|}$.

Then $\{\underset{\sim}{b}_1, \dots, \underset{\sim}{b}_k, \underset{\sim}{b}_{k+1}\}$ is an orthonormal basis for W_{k+1}.

Continue this process until $k+1 = n$, and then $\{\underset{\sim}{b}_1, \dots, \underset{\sim}{b}_n\}$ becomes an orthormal basis for V.

Example: Find an orthonormal basis for the hyperplane W in $\mathbb{R}^4$ satisfying the equation

$$x + y + z + w = 0,$$

and use it to find the closest point on W to

$$\underset{\sim}{v} = (1,2,3,4),$$

and the shortest distance from $\underset{\sim}{v}$ to W.

Solution: Observe that

$$W = \{(x,y,z,w) \mid x+y+z+w=0\}$$

$$= \{(-y-z-w, y, z, w) \mid y,z,w \in \mathbb{R}\}$$

$$= \{y(-1,1,0,0) + z(-1,0,1,0) + w(-1,0,0,1) \mid y,z,w \in \mathbb{R}\}$$

Example: Find an orthonormal basis for the hyperplane W in $\mathbb{R}^4$ satisfying the equation

$$x + y + z + w = 0,$$

and use it to find the closest point on W to

$$\underset{\sim}{v} = (1, 2, 3, 4),$$

and the shortest distance from $\underset{\sim}{v}$ to W.

Solution: Observe that

$$W = \{ (x, y, z, w) \mid x + y + z + w = 0 \}$$

$$= \{ (-y-z-w, y, z, w) \mid y, z, w \in \mathbb{R} \}$$

$$= \{ y(-1,1,0,0) + z(-1,0,1,0) + w(-1,0,0,1) \mid y, z, w \in \mathbb{R} \}$$

so that W is spanned by

$$\underset{\sim}{v}_1 = (-1,1,0,0), \quad \underset{\sim}{v}_2 = (-1,0,1,0), \quad \underset{\sim}{v}_3 = (-1,0,0,1).$$

It is easy to check that $B = \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$ is LI,

so B forms a basis for W.

We apply the Gram-Schmidt process:

Step (1): Put

$$\underset{\sim}{b}_1 = \hat{\underset{\sim}{v}}_1 = \frac{\underset{\sim}{v}_1}{\|\underset{\sim}{v}_1\|} = \frac{1}{\sqrt{2}}(-1,1,0,0)$$

and let

W_1 = subspace spanned by $\underset{\sim}{b}_1$ = subspace spanned by $\underset{\sim}{v}_1$.

Step (2): Put

$$\underset{\sim}{w}_2 = \underset{\sim}{v}_2 - \mathrm{proj}_{W_1}\underset{\sim}{v}_2 = \underset{\sim}{v}_2 - (\underset{\sim}{v}_2 \cdot \underset{\sim}{b}_1)\,\underset{\sim}{b}_1$$

$$= (-1,0,1,0) - \left((-1,0,1,0) \cdot \tfrac{1}{\sqrt{2}}(-1,1,0,0)\right)\tfrac{1}{\sqrt{2}}(-1,1,0,0)$$

$$= (-1,0,1,0) - \tfrac{1}{2}(-1,1,0,0) = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0\right).$$

Step (2): Put

$$\underset{\sim}{w}_2 = \underset{\sim}{v}_2 - \text{proj}_{W_1} \underset{\sim}{v}_2 = \underset{\sim}{v}_2 - (\underset{\sim}{v}_2 \cdot \underset{\sim}{b}_1)\, \underset{\sim}{b}_1$$

$$= (-1,0,1,0) - \left((-1,0,1,0) \cdot \tfrac{1}{\sqrt{2}}(-1,1,0,0) \right) \tfrac{1}{\sqrt{2}}(-1,1,0,0)$$

$$= (-1,0,1,0) - \tfrac{1}{2}(-1,1,0,0) = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0\right),$$

and

$$\underset{\sim}{b}_2 = \hat{\underset{\sim}{w}}_2 = \frac{\underset{\sim}{w}_2}{\|\underset{\sim}{w}_2\|} = \frac{(-1,-1,2,0)}{\|(-1,-1,2,0)\|} = \frac{1}{\sqrt{6}}(-1,-1,2,0).$$

Then $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2\}$ is an orthonormal basis for

W_2 = subspace spanned by $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2\}$ = subspace spanned by $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$.

Step (3): Put

$$\underset{\sim}{w}_3 = \underset{\sim}{v}_3 - \text{proj}_{W_2} \underset{\sim}{v}_3 = \underset{\sim}{v}_3 - (\underset{\sim}{v}_3 \cdot \underset{\sim}{b}_1)\, \underset{\sim}{b}_1 - (\underset{\sim}{v}_3 \cdot \underset{\sim}{b}_2)\, \underset{\sim}{b}_2$$

Step (3): Put

$$\underset{\sim}{w}_3 = \underset{\sim}{v}_3 - \text{proj}_{W_2}\underset{\sim}{v}_3 = \underset{\sim}{v}_3 - (\underset{\sim}{v}_3 \cdot \underset{\sim}{b}_1)\underset{\sim}{b}_1 - (\underset{\sim}{v}_3 \cdot \underset{\sim}{b}_2)\underset{\sim}{b}_2$$

$$= (-1,0,0,1) - \left((-1,0,0,1)\cdot \tfrac{1}{\sqrt{2}}(-1,1,0,0)\right)\tfrac{1}{\sqrt{2}}(-1,1,0,0)$$

$$- \left((-1,0,0,1)\cdot \tfrac{1}{\sqrt{6}}(-1,-1,2,0)\right)\tfrac{1}{\sqrt{6}}(-1,-1,2,0)$$

$$= (-1,0,0,1) - \tfrac{1}{2}(-1,1,0,0) - \tfrac{1}{6}(-1,-1,2,0) = \left(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, 1\right),$$

and

$$\underset{\sim}{b}_3 = \hat{\underset{\sim}{w}}_3 = \frac{\underset{\sim}{w}_3}{\|\underset{\sim}{w}_3\|} = \frac{(-1,-1,-1,3)}{\|(-1,-1,-1,3)\|} = \frac{1}{2\sqrt{3}}(-1,-1,-1,3),$$

Then

$$\boxed{\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\} = \left\{\tfrac{1}{\sqrt{2}}(-1,1,0,0), \tfrac{1}{\sqrt{6}}(-1,-1,2,0), \tfrac{1}{2\sqrt{3}}(-1,-1,-1,3)\right\}}$$

is an orthonormal basis for W.

Then

$$\boxed{\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\} = \left\{\tfrac{1}{\sqrt{2}}(-1,1,0,0), \tfrac{1}{\sqrt{6}}(-1,-1,2,0), \tfrac{1}{2\sqrt{3}}(-1,-1,-1,3)\right\}}$$

is an orthonormal basis for W.

Thus the closest point on W to $\underset{\sim}{v} = (1,2,3,4)$ is

$$\mathrm{proj}_W \underset{\sim}{v} = (\underset{\sim}{v}\cdot\underset{\sim}{b}_1)\underset{\sim}{b}_1 + (\underset{\sim}{v}\cdot\underset{\sim}{b}_2)\underset{\sim}{b}_2 + (\underset{\sim}{v}\cdot\underset{\sim}{b}_3)\underset{\sim}{b}_3$$

$$= \left((1,2,3,4)\cdot\tfrac{1}{\sqrt{2}}(-1,1,0,0)\right)\tfrac{1}{\sqrt{2}}(-1,1,0,0)$$

$$+ \left((1,2,3,4)\cdot\tfrac{1}{\sqrt{6}}(-1,-1,2,0)\right)\tfrac{1}{\sqrt{6}}(-1,-1,2,0)$$

$$+ \left((1,2,3,4)\cdot\tfrac{1}{2\sqrt{3}}(-1,-1,-1,3)\right)\tfrac{1}{2\sqrt{3}}(-1,-1,-1,3)$$

$$= \tfrac{1}{2}(-1,1,0,0) + \tfrac{3}{6}(-1,-1,2,0) + \tfrac{6}{12}(-1,-1,-1,3)$$

$$= \left(-\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}\right).$$

Then

$$\boxed{\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\} = \left\{\tfrac{1}{\sqrt{2}}(-1,1,0,0),\ \tfrac{1}{\sqrt{6}}(-1,-1,2,0),\ \tfrac{1}{2\sqrt{3}}(-1,-1,-1,3)\right\}}$$

is an orthonormal basis for W.

Thus the closest point on W to $\underset{\sim}{v} = (1,2,3,4)$ is

$$\text{proj}_W \underset{\sim}{v} = (\underset{\sim}{v}\cdot\underset{\sim}{b}_1)\underset{\sim}{b}_1 + (\underset{\sim}{v}\cdot\underset{\sim}{b}_2)\underset{\sim}{b}_2 + (\underset{\sim}{v}\cdot\underset{\sim}{b}_3)\underset{\sim}{b}_3$$

$$= \left(-\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}\right).$$

The distance from $\underset{\sim}{v}$ to W is

$$\left\| \underset{\sim}{v} - \left(-\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}\right)\right\| = \left\|(1,2,3,4) - \left(-\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}\right)\right\|$$

$$= \left\|\left(\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}\right)\right\| = \tfrac{5}{2}\|(1,1,1,1)\|$$

$$= \tfrac{5}{2}\sqrt{1^2+1^2+1^2+1^2} = \tfrac{5}{2}\sqrt{4} = \underline{\underline{5}}.$$

Then

$$\boxed{\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\} = \left\{\frac{1}{\sqrt{2}}(-1,1,0,0), \frac{1}{\sqrt{6}}(-1,-1,2,0), \frac{1}{2\sqrt{3}}(-1,-1,-1,3)\right\}}$$

is an orthonormal basis for W.

Thus the closest point on W to $\underset{\sim}{v} = (1,2,3,4)$ is

$$\text{proj}_W \underset{\sim}{v} = (\underset{\sim}{v}\cdot\underset{\sim}{b}_1)\underset{\sim}{b}_1 + (\underset{\sim}{v}\cdot\underset{\sim}{b}_2)\underset{\sim}{b}_2 + (\underset{\sim}{v}\cdot\underset{\sim}{b}_3)\underset{\sim}{b}_3$$

$$= \left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right).$$

The distance from $\underset{\sim}{v}$ to W is

$$\left\|\underset{\sim}{v} - \left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)\right\| = \underline{\underline{5}}.$$

Check: $\underset{\sim}{v} - \left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right) = \left(5/2, 5/2, 5/2, 5/2\right)$

should be orthogonal to $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3$.

Then

$$\boxed{\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \left\{ \tfrac{1}{\sqrt{2}}(-1,1,0,0),\ \tfrac{1}{\sqrt{6}}(-1,-1,2,0),\ \tfrac{1}{2\sqrt{3}}(-1,-1,-1,3) \right\}}$$

is an orthonormal basis for W.

<u>**Check**</u>: $\mathbf{v} - \left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right) = \left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}\right)$

should be orthogonal to $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$.

$$\left(\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}\right) \cdot \tfrac{1}{\sqrt{2}}(-1,1,0,0) = \tfrac{1}{\sqrt{2}}\left(-\tfrac{5}{2} + \tfrac{5}{2}\right) = 0$$

$$\left(\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}\right) \cdot \tfrac{1}{\sqrt{6}}(-1,-1,2,0) = \tfrac{1}{\sqrt{6}}\left(-\tfrac{5}{2} - \tfrac{5}{2} + 5\right) = 0$$

$$\left(\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}\right) \cdot \tfrac{1}{2\sqrt{3}}(-1,-1,-1,3) = \tfrac{1}{2\sqrt{3}}\left(-\tfrac{5}{2} - \tfrac{5}{2} - \tfrac{5}{2} + \tfrac{15}{2}\right) = 0$$

Quiz Practice

Q11/ Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(x,y) = (-x-2y, 2y-2x)$.

Find an ordered basis B for $\mathbb{R}^2$ such that

$$[L]_B^B = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}.$$

(a) $B = \{(2,-1), (1,2)\}$ (b) $B = \{(3,1), (1,-1)\}$

(c) $B = \{(2,1), (-1,2)\}$ (d) $B = \{(-1,-2), (1,-2)\}$

(e) $B = \{(-2,0), (0,3)\}$

Solution : (a) $L(2,-1) = (-2+2, -2-4) = (0,-6)$

$$\neq -2(2,-1)$$

Quiz Practice

Q11/ Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(x,y) = (-x-2y, 2y-2x)$.

Find an ordered basis B for $\mathbb{R}^2$ such that

$$[L]_B^B = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}.$$

(a) $B = \{(2,-1), (1,2)\}$

(b) $B = \{(3,1), (1,-1)\}$

(c) $B = \{(2,1), (-1,2)\}$

(d) $B = \{(-1,-2), (1,-2)\}$

(e) $B = \{(-2,0), (0,3)\}$

Solution : (b) $L(3,1) = (-3-2, 2-6) = (-5,-4)$

$\neq -2(3,1)$

Quiz Practice

Q11/ Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(x,y) = (-x-2y, 2y-2x)$.
Find an ordered basis B for $\mathbb{R}^2$ such that

$$[L]_B^B = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}.$$

(a) $B = \{(2,-1), (1,2)\}$

(b) $B = \{(3,1), (1,-1)\}$

✓ (c) $B = \{(2,1), (-1,2)\}$

(d) $B = \{(-1,-2), (1,-2)\}$

(e) $B = \{(-2,0), (0,3)\}$

Solution: (c) ✓

$L(2,1) = (-2-2, 2-4) = (-4,-2)$

$= -2(2,1)$ ✓

$L(-1,2) = (1-4, 4+2) = (-3,6) = 3(-1,2)$ ✓

Q12/ Let $V = \langle B \rangle$ where $B = \{e^{-x}, xe^{-x}, x^2e^{-x}\}$.

Find $[D]_B^B$ where $D : V \to V$, $f \mapsto f'$

(the differential operator).

<u>Solution</u> :

$$D(e^{-x}) = -e^{-x} = (-1)e^{-x} + (0)\, xe^{-x} + (0)\, x^2e^{-x}$$

$$D(xe^{-x}) = e^{-x} - xe^{-x} = (1)e^{-x} + (-1)xe^{-x} + (0)x^2e^{-x}$$

$$D(x^2e^{-x}) = 2xe^{-x} - x^2e^{-x} = (0)e^{-x} + (2)xe^{-x} + (-1)x^2e^{-x}$$

So

$$[D]_B^B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

Q13/ Let $V = \langle B \rangle$ where $B = \{\sinh x, \cosh x\}$.
Find $[D]_B^B$ where $D : V \to V$, $f \mapsto f'$
(the differential operator).

<u>Solution</u>:

$$D(\sinh x) = \cosh x = (0)\sinh x + (1)\cosh x$$
$$D(\cosh x) = \sinh x = (1)\sinh x + (0)\cosh x$$

so

$$[D]_B^B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Q19/ Consider the permutations

$$\alpha = (1\,3\,2)(4\,6)(5\,8)(7\,9), \quad \beta = (1\,4\,2)(8\,6)(3\,7)(5\,9).$$

How many permutations γ of $\{1,2,3,4,5,6,7,8,9\}$ exist with the property

$$\beta = \gamma^{-1}\alpha\gamma.$$

<u>Solution</u>

$$\alpha = (1\,3\,2)(4\,6)(5\,8)(7\,9)$$

position $\quad \beta = (a\,b\,c)\,(d\,e)\,(f\,g)\,(h\,i)$

where

$(a\,b\,c) = (1\,4\,2)$ in 3 possible ways,

$(d\,e)(f\,g)(h\,i) = (8\,6)(3\,7)(5\,9)$ in $3! \times 2 \times 2 \times 2$

$= 48$ possible ways.

Q19/ Consider the permutations

$$\alpha = (1\,3\,2)(4\,6)(5\,8)(7\,9), \quad \beta = (1\,4\,2)(8\,6)(3\,7)(5\,9).$$

How many permutations γ of $\{1,2,3,4,5,6,7,8,9\}$ exist with the property

$$\beta = \gamma^{-1}\alpha\gamma.$$

Solution

$$\alpha = (1\,3\,2)(4\,6)(5\,8)(7\,9)$$

position $\quad \beta = (a\,b\,c)\,(d\,e)\,(f\,g)\,(h\,i)$

where

$(a\,b\,c) = (1\,4\,2)$ in 3 possible ways,

$(de)(fg)(hi) = (8\,6)(3\,7)(5\,9)$ in $3! \times 2 \times 2 \times 2$

$= 48$ possible ways.

Thus there are

$3 \times 48 = 144$ possibilities for γ.

Q20/ $\{a, b, c\} = \{1, 2, 3\}$ and, from the 8-puzzle,

1	2	3
4	5	6
7	8	

the following configuration has been reached:

a	b	c
6		8
7	5	4

Q20/ $\{a, b, c\} = \{1, 2, 3\}$ and, from the 8-puzzle,

1	2	3
4	5	6
7	8	

the following configuration has been reached:

a	b	c
6		8
7	5	4

Which is true?

(a) $a=1 \Rightarrow b=3$

(b) $b=1 \Rightarrow c=2$

(c) $a=2 \Rightarrow c=3$

(d) $c=3 \Rightarrow b=1$

(e) $c=1 \Rightarrow b=2$

Q20/

a	b	c
6		8
7	5	4

Which is true?

(a) $a=1 \Rightarrow b=3$

(b) $b=1 \Rightarrow c=2$

(c) $a=2 \Rightarrow c=3$

(d) $c=3 \Rightarrow b=1$

(e) $c=1 \Rightarrow b=2$

Solution :

(a)

1	3	2
6		8
7	5	4

(b)

3	1	2
6		8
7	5	4

(c)

2	1	3
6		8
7	5	4

(d)

2	1	3
6		8
7	5	4

(e)

3	2	1
6		8
7	5	4

Q20/

a	b	c
6		8
7	5	4

Which is true?

(a) $a=1 \Rightarrow b=3$

(b) $b=1 \Rightarrow c=2$

(c) $a=2 \Rightarrow c=3$

(d) $c=3 \Rightarrow b=1$

(e) $c=1 \Rightarrow b=2$

Solution:

(a)

1	3	2
6		8
7	5	4

(b)

3	1	2
6		8
7	5	4

since (a), (c), (d), (e) are all an even permutation from each other

(c)

2	1	3
6		8
7	5	4

(d)

2	1	3
6		8
7	5	4

(e)

3	2	1
6		8
7	5	4

Q20/

a	b	c
6		8
7	5	4

Which is true?

(a) $a=1 \Rightarrow b=3$

(b) $b=1 \Rightarrow c=2$

(c) $a=2 \Rightarrow c=3$

(d) $c=3 \Rightarrow b=1$

(e) $c=1 \Rightarrow b=2$

Solution:

(b)

3	1	2
6		8
7	5	4

✓

since (a), (c), (d), (e) are all an even permutation from each other

Check:

3	1	2
6	9	8
7	5	4

corresponds to

$(1\ 2\ 3)(4\ 9\ 5\ 8\ 6)$

which is even ✓✓