MATH 2022 Linear and Abstract Algebra

LECTURE 05 Wednesday 04/03/2020

Transpose of a matrix :

If M is an $m \times n$ matrix then M^T, the **transpose** of M, is the $n \times m$ matrix obtained by interchanging rows and columns.

e.g. Working over $\mathbb{R}$, put

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \underset{\sim}{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Then

$$M^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \quad \underset{\sim}{v}^T = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

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$$M\underset{\sim}{v} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

and

$$\underset{\sim}{v}^T M^T = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 11 \end{bmatrix} = (M\underset{\sim}{v})^T.$$

In general, when sums and products are defined,

$$\boxed{(A+B)^T = A^T + B^T, \quad (A^T)^T = A, \quad (AB)^T = B^T A^T.}$$

good exercise in Sigma notation

Solving systems of equations:

All the familiar techniques carry over to matrices where entries come from any fixed field $\mathbb{F}$.

A system of m linear equations in n variables $x_1, \dots, x_n$ has the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \quad &\quad\ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

where a_{ij}, b_i are constants from $\mathbb{F}$.

The system has **augmented matrix**

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

homogeneous when $b_1 = \cdots = b_m = 0$

which is an abbreviation for the matrix equation

$$M\underset{\sim}{x} = \underset{\sim}{b}$$

where

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

To solve the system:

- perform Gaussian elimination using elementary row operations (e.r.o.'s)
- to convert the augmented matrix into row echelon form
- and then apply back substitution
- assigning parameters to non-leading variables.

Three types of e.r.o.'s :

(1) interchanging ith and jth rows, denoted by

$$R_i \longleftrightarrow R_j \,,$$

(2) multiplying the ith row by a nonzero constant $\lambda \in F$, denoted by

$$R_i \longrightarrow \lambda R_i \,,$$

(3) adding a multiple of the jth row to the ith row, denoted by

$$R_i \longrightarrow R_i + \lambda R_j \,.$$

For a matrix to be in row echelon form:

(a) rows of zeros appear at the bottom,

(b) first nonzero (leading) entries of consecutive rows appear further to the right,

(c) leading entries of rows are equal to 1.

in practice, (c) is often relaxed

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To be in reduced row echelon form, we also require

(d) entries above (and below) leading entries are zero.

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back substitution can be applied

simplifies if (d) holds

Theorem (difficult): A given matrix can be row reduced to one and only one matrix in reduced row echelon form.

(Row echelon forms that are not reduced need not be unique.)

A system is **inconsistent** (has no solution) iff at some stage during row reduction a row of zeros appears, followed by nonzero k to the right

$$0 \quad 0 \quad \dots \quad 0 \;\Big|\; k$$

Example: Solve the following over $\mathbb{R}$ and $\mathbb{Z}_7$:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 2$$
$$x_1 - x_2 + x_3 \qquad\quad - x_5 = 0$$
$$x_1 + 5x_2 + x_3 + 3x_4 + 3x_5 = 3$$

Solution:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & 5 & 1 & 3 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & -1 & -2 & -2 \\ 0 & 4 & 0 & 2 & 2 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 & -3 \end{array}\right] \begin{array}{l} R_2 \to -\frac{1}{2} R_2 \\ R_3 \to R_3 + 2R_2 \end{array}$$

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$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3/2 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to -\frac{1}{2} R_3 \end{array}$$

in reduced row echelon form

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & 3/2 \end{array}\right] \quad R_2 \to R_2 - R_3$$

Solution:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & 5 & 1 & 3 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & 3/2 \end{array}\right]$$

giving an equivalent system:

$$\boxed{\begin{aligned} x_1 + x_3 + \tfrac{1}{2}x_4 &= 1 \\ x_2 + \tfrac{1}{2}x_4 &= -1/2 \\ x_5 &= 3/2 \end{aligned}}$$

The leading variables are x_1, x_2, x_5.

The non-leading variables are x_3, x_4.

Solution :

$$\boxed{\begin{array}{lllllll} x_1 & & + x_3 & + \frac{1}{2}x_4 & & = & 1 \\ & x_2 & & + \frac{1}{2}x_4 & & = & -1/2 \\ & & & & x_5 & = & 3/2 \end{array}}$$

The leading variables are x_1, x_2, x_5.

The non-leading variables are x_3, x_4.

Back substitution : $x_5 = 3/2$, $x_4 = t$, $x_3 = s$,

$$x_2 = -1/2 - \frac{1}{2}x_4 = -\frac{1}{2} - t/2,$$

$$x_1 = 1 - x_3 - \frac{1}{2}x_4 = 1 - s - t/2.$$

Solution :

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Solution over $\mathbb{R}$:

$$x_1 = 1 - s - \frac{t}{2}, \quad x_2 = -\frac{1}{2} - \frac{t}{2}, \quad x_3 = s, \quad x_4 = t, \quad x_5 = \frac{3}{2}$$

where $s, t \in \mathbb{R}$.

Solution set : $\{ (1 - s - t/2, -\tfrac{1}{2} - t/2, s, t, 3/2) \mid s, t \in \mathbb{R} \}$.

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Solution over $\mathbb{Z}_7$: all fractions involved dividing by 2, sensible in $\mathbb{Z}_7$, so same solution description holds over $\mathbb{Z}_7$, but simplifies because $\frac{1}{2} = 4$ in $\mathbb{Z}_7$.

<u>Solution</u> :

$\boxed{\begin{array}{l}\text{Solution set over } \mathbb{R} : \\ \left\{ \left(1 - s - \tfrac{t}{2},\ -\tfrac{1}{2} - \tfrac{t}{2},\ s,\ t,\ \tfrac{3}{2}\right) \mid s, t \in \mathbb{R} \right\}.\end{array}}$

Solution over $\mathbb{Z}_7$: all fractions involved dividing by 2, sensible in $\mathbb{Z}_7$, so same solution description holds over $\mathbb{Z}_7$, but simplifies because $\frac{1}{2} = 4$ in $\mathbb{Z}_7$.

$\boxed{\begin{array}{l}\text{Solution set over } \mathbb{Z}_7 : \\ \left\{ \left(1 + 6s + 3t,\ 3 + 3t,\ s,\ t,\ 5\right) \mid s, t \in \mathbb{Z}_7 \right\}\end{array}}$

The solution set is <u>infinite</u> over $\mathbb{R}$, but has $7^2 = 49$ solutions over $\mathbb{Z}_7$.