MATH 2022 Linear and Abstract Algebra

LECTURE 07 Tuesday 10/03/2020

Elementary matrices & inversion

- relating elementary row operations (e.r.o.'s) to multiplication by elementary matrices

  "atoms" for matrix "decompositions"

- exploiting inverses of elementary matrices to develop a criterion and algorithm for finding inverses of any square matrix
- finding LDU and LU - decompositions of a matrix

Elementary matrices & inversion

- relating elementary row operations (e.r.o.'s) to multiplication by elementary matrices

"atoms" for matrix "decompositions"

- finding LDU and LU - decompositions of a matrix

lower-diagonal-upper

lower-upper

triangular matrices

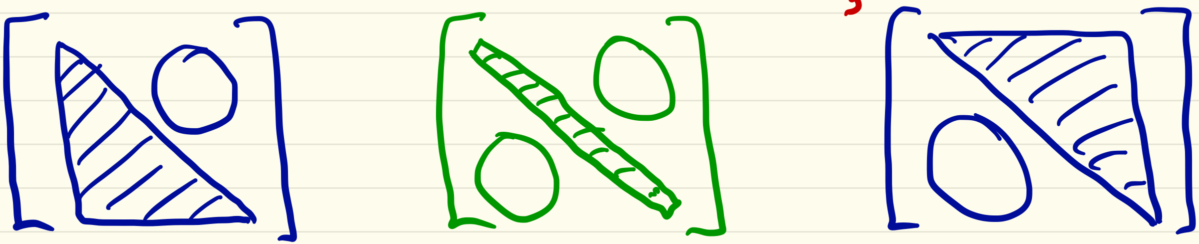
Elementary matrices & inversion

- finding **LDU** and **LU** - decompositions of a matrix

lower-diagonal-upper

lower-upper

triangular matrices



Fact: Elementary row operations (e.r.o.'s) can be achieved by premultiplication by elementary matrices.

multiplication on the left

postmultiplication (multiplication on the right) leads to a theory of elementary column operations

e.r.o.	achieved by **elementary matrix**
$R_i \leftrightarrow R_j$ $(i \neq j)$	see below

$$
\begin{array}{cc}
 & \begin{matrix} & & & i & & & & j & & & \end{matrix} \\
\begin{matrix} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{matrix} &
\begin{bmatrix}
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & & & 0 & \cdots & \cdots & \cdots & 1 & & & \\
 & & & \vdots & 1 & & & \vdots & & & \\
 & & & \vdots & & \ddots & & \vdots & & & \\
 & & & \vdots & & & 1 & \vdots & & & \\
 & & & 1 & \cdots & \cdots & \cdots & 0 & & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{bmatrix}
\end{array}
$$

result of applying e.r.o. to the identity matrix I_n

e.g.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & -1 \\ 0 & 0 & 4 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

e.r.o.	achieved by elementary matrix
$R_i \longrightarrow \lambda R_i$ $(\lambda \neq 0)$	$\begin{array}{cc} & \begin{array}{ccccccc} & & & i & & & \end{array} \\ i & \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \lambda & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \end{array}$

result of applying e.r.o. to the identity matrix I_n

e.g.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to 4R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & -1 \\ 1 & 2 & 0 & -3 \\ -1 & 6 & 0 & 2 \\ 0 & 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & -1 \\ 4 & 8 & 0 & -12 \\ -1 & 6 & 0 & 2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ \frac{g}{2} & \frac{h}{2} & \frac{i}{2} \end{bmatrix}$$

e.r.o.	achieved by elementary matrix
$R_i \to R_i + \lambda R_j$ $(i \neq j)$	see below

$$
\begin{array}{cc}
 & \begin{matrix} & & & i & & & & j & & & \end{matrix} \\
\begin{matrix} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{matrix} &
\begin{bmatrix}
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & & & 1 & \cdots & \cdots & \cdots & \lambda & & & \\
 & & & \vdots & 1 & & & \vdots & & & \\
 & & & \vdots & & \ddots & & \vdots & & & \\
 & & & \vdots & & & 1 & \vdots & & & \\
 & & & 0 & \cdots & \cdots & \cdots & 1 & & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{bmatrix}
\end{array}
$$

$i < j$

result of applying e.r.o. to the identity matrix I_n

e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 3R_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 6 \\ 3 & 4 & 0 \\ -1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 0 \\ 3 & 4 & 0 \\ -1 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 6 & 3 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

e.r.o.	achieved by elementary matrix
$R_i \to R_i + \lambda R_j$ $(i \neq j)$	see below

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccc} & & & j & & & & i & & & \end{array} \\
\begin{array}{c} \\ \\ \\ j \\ \\ \\ \\ i \\ \\ \\ \\ \end{array} &
\begin{bmatrix}
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & & & 1 & & \cdots & & 0 & & & \\
 & & & \vdots & 1 & & & \vdots & & & \\
 & & & \vdots & & \ddots & & \vdots & & & \\
 & & & \vdots & & & 1 & \vdots & & & \\
 & & & \lambda & & \cdots & & 1 & & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{bmatrix}
\end{array}
$$

$i > j$

result of applying e.r.o. to the identity matrix I_n

e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 3R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 6 \\ 3 & 4 & 0 \\ -1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 3 & 4 & 0 \\ 5 & -1 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 6 & 3 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 & 3 \\ 0 & 1 & 4 & 1 \\ 0 & -2 & -6 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

e.r.o.	achieved by elementary matrix	
$R_i \longleftrightarrow R_j$ $(i \neq j)$	$\begin{array}{cc} & \begin{matrix} & & i & & j & & \end{matrix} \\ \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} & \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \end{array}$	
$R_i \longrightarrow \lambda R_i$ $(\lambda \neq 0)$	$\begin{array}{cc} & \begin{matrix} & & i & & \end{matrix} \\ \begin{matrix} \\ \\ i \\ \\ \\ \end{matrix} & \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \end{array}$	
$R_i \to R_i + \lambda R_j$ $(i \neq j)$	$\begin{array}{cc} & \begin{matrix} & & i & & j & & \end{matrix} \\ \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} & \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & \cdots & \lambda & & \\ & & \vdots & \ddots & \vdots & & \\ & & 0 & \cdots & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \end{array}$ $i<j$	$\begin{array}{cc} & \begin{matrix} & & j & & i & & \end{matrix} \\ \begin{matrix} \\ \\ j \\ \\ i \\ \\ \\ \end{matrix} & \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & \cdots & 0 & & \\ & & \vdots & \ddots & \vdots & & \\ & & \lambda & \cdots & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \end{array}$ $i>j$

result of applying e.r.o. to the identity matrix I_n

$\boxed{\text{Simple fact: If } E \text{ is an elementary matrix then } E \text{ is invertible and } E^{-1} \text{ corresponds to the inverse e.r.o. that undoes the e.r.o. corresponding to } E.}$

e.g. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ corresponds to $R_1 \to R_1 + 2R_2$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ " " $R_1 \to R_1 - 2R_2$

e.g. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ corresponds to $R_1 \to R_1 + 2R_2$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ " " $R_1 \to R_1 - 2R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$ " " $R_3 \to R_3 - 4R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$ " " $R_3 \to R_3 + 4R_2$

e.g.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ corresponds to } R_1 \to R_1 + 2R_2$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \quad \text{" \quad "} \quad R_1 \to R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad \text{" \quad "} \quad R_3 \to R_3 - 4R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \quad \text{" \quad "} \quad R_3 \to R_3 + 4R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{" \quad "} \quad R_2 \to 4R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{" \quad "} \quad R_2 \to \frac{1}{4} R_2$$

e.g.

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ corresponds to $R_1 \to R_1 + 2R_2$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ " " $R_1 \to R_1 - 2R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$ " " $R_3 \to R_3 - 4R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$ " " $R_3 \to R_3 + 4R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ " " $R_2 \to 4R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ " " $R_2 \to \frac{1}{4} R_2$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ " " $R_1 \leftrightarrow R_3$

self-inverse

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ " " $R_1 \leftrightarrow R_3$

Theorem: Let A be a square matrix. Then A is invertible iff A can be row reduced to I, the identity matrix.

Proof: $(\Leftarrow)$ Suppose that A can be row reduced to I, say using e.r.o.'s corresponding to elementary matrices

$$E_1, E_2, \ldots, E_k.$$

Then

$$\underbrace{E_k E_{k-1} \ldots E_2 E_1 A}_{} = I.$$

note reverse order of premultiplication

Then

$$A = I A$$

$$= (E_k E_{k-1} \cdots E_2 E_1)^{-1} (E_k E_{k-1} \cdots E_2 E_1) A$$

$$= E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1} \quad I$$

$$= E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1} ,$$

so that A is invertible, being a product of invertible matrices.

Then
$$
\begin{aligned}
A &= I\,A \\
&= (E_k E_{k-1} \cdots E_2 E_1)^{-1} (E_k E_{k-1} \cdots E_2 E_1)\, A \\
&= E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}\, I \\
&= E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1},
\end{aligned}
$$
so that A is invertible, being a product of invertible matrices.

Thus
$$
\begin{aligned}
A^{-1} &= (E_1^{-1} \cdots E_k^{-1})^{-1} = (E_k^{-1})^{-1} \cdots (E_1^{-1})^{-1} \\
&= E_k \cdots E_1,
\end{aligned}
$$

$$\boxed{A^{-1} = E_k \cdots E_1}.$$

product of elementary matices in reverse order

Theorem: Let A be a square matrix. Then A is invertible iff A can be row reduced to I, the identity matrix.

Proof: ($\Rightarrow$) Suppose conversely that A is invertible. There exist e.r.o.'s that transform A into some matrix M in reduced row echelon form. Thus there are some elementary matrices

$$E_1, \dots, E_k$$

such that

$$M = E_k \cdots E_1 A.$$

In particular, M is invertible

(being a product of invertible matrices),

so cannot have a row of zeros.

easy exercise

But the only square matrix in reduced row echelon form <u>without</u> a row of zeros is

$$I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}.$$

Thus $M = I$.

$\square$

The proof yields an algorithm for finding A^{-1}:

$[A \mid I] \sim [E_1 A \mid E_1]$ using e.r.o. for E_1

$\sim [E_2 E_1 A \mid E_2 E_1]$ " " " E_2

$\sim \ \cdots$

$\sim [\underbrace{E_k E_{k-1} \cdots E_2 E_1 A} \mid \underbrace{E_k E_{k-1} \cdots E_2 E_1}]$ " " " E_k

if reduced row echelon form is I

then contents yield $A^{-1} = E_k \cdots E_1$

Example: Invert $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ over $\mathbb{Z}_5$.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 \to R_2 - R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 4 & 1 & 0 \\ 0 & 0 & 2 & 4 & 1 & 1 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 4 & 0 \\ 0 & 0 & 1 & 2 & 3 & 3 \end{array}\right] \quad \begin{array}{l} R_2 \to -R_2 \\ R_3 \to 3R_3 \end{array}$$

Example: Invert $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ over $\mathbb{Z}_5$.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 4 & 0 \\ 0 & 0 & 1 & 2 & 3 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 1 & 4 & 0 \\ 0 & 0 & 1 & 2 & 3 & 3 \end{array}\right] \quad R_1 \to R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & 2 \\ 0 & 1 & 0 & 3 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 & 3 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 + R_3 \end{array}$$

Example: Invert $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ over $\mathbb{Z}_5$.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & 2 \\ 0 & 1 & 0 & 3 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 & 3 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} 3 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}.$$

Check: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

Consider the following elementary matrices:

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ corresponding to $R_2 \to R_2 - R_1$

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ " " $R_3 \to R_3 + R_2$

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ " " $R_2 \to -R_2$

$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ " " $R_3 \to 3R_3$

$E_5 = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ " " $R_1 \to R_1 - R_2$

$E_6 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ " " $R_1 \to R_1 - R_3$

$E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ " " $R_2 \to R_2 + R_3$

Then

$$E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I$$

so

$$A = (E_1^{-1} E_2^{-1})(E_3^{-1} E_4^{-1})(E_5^{-1} E_6^{-1} E_7^{-1})$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \right) \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \right) \left(\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \right)$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}}_{\text{lower triangular with 1's down diagonal}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{\text{diagonal}} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{upper triangular with 1's above diagonal}}$$

check!

Then

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}}_{\substack{\text{lower triangular} \\ \text{with 1's down diagonal}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{\text{diagonal}} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{upper triangular} \\ \text{with 1's above diagonal}}}$$

known as an LDU-decomposition.

One can combine the middle diagonal matrix with either the first or third matrices, in different ways, to get LU-decompositions of the matrix.

Then

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}}_{\text{combining 1st \& 2nd}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{\text{combining 2nd \& 3rd}}$$

One can combine the middle diagonal matrix with either the first or third matrices, in different ways, to get LU-decompositions of the matrix.