MATH 2022 Linear and Abstract Algebra

LECTURE 09 Thursday 12/03/2020

Odd and even permutations

- permutations of a finite set can be either **even** or **odd**, but not both.

related to **even** integers

related to **odd** integers

tricky theorem

- many surprising applications

Transpositions:

A transposition is a permutation that interchanges two distinct letters a and b, leaving all other letters unchanged, described in cycle notation by

$$(a\ b).$$

Fact: All cycles are products (composites) of transpositions, since

$$(a_1\ a_2 \cdots a_n) = (a_1 a_2)(a_1 a_3) \cdots (a_1 a_n).$$

e.g. $(1\ 2\ 3) = (1\ 2)(1\ 3)$,

$(1\ 2\ 3) = (2\ 3\ 1) = (2\ 3)(2\ 1)$,

$(1\ 2\ 3\ 4) = (1\ 2)(1\ 3)(1\ 4)$,

$(5\ 6\ 3\ 1\ 4\ 2) = (5\ 6)(5\ 3)(5\ 1)(5\ 4)(5\ 2)$,

$\|$

$(6\ 3\ 1\ 4\ 2\ 5) = (6\ 3)(6\ 1)(6\ 4)(6\ 2)(6\ 5)$.

<u>Corollary</u>: Every permutation of a finite set is a product (composite) of transpositions.

e.g. $(1\ 2\ 3\ 4)(5\ 6\ 7) = (1\ 2)(1\ 3)(1\ 4)(5\ 6)(5\ 7)$,

$(2\ 6\ 4)(3\ 2\ 1)(4\ 7\ 8\ 9\ 2)$

$= (2\ 6)(2\ 4)(3\ 2)(3\ 1)(4\ 7)(4\ 8)(4\ 9)(4\ 2)$.

Call a permutation <u>even</u> or <u>odd</u> according to whether it is a product of an even or an odd number of transpositions respectively.

e.g. $(1\ 2\ 3) = (1\ 2)(1\ 3)$ is even,

$(1\ 2\ 3\ 4) = (1\ 2)(1\ 3)(1\ 4)$ is odd,

$1 = (2\ 3)(2\ 3)$ is even.

e.g. $(1\ 2\ 3\ 4)(5\ 6\ 7) = (1\ 2)(1\ 3)(1\ 4)(5\ 6)(5\ 7)$, odd

$(2\ 6\ 4)(3\ 2\ 1)(4\ 7\ 8\ 9\ 2)$ even

$= (2\ 6)(2\ 4)(3\ 2)(3\ 1)(4\ 7)(4\ 8)(4\ 9)(4\ 2)$.

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The single cycle

$$(a_1\ a_2 \dots a_n) \quad \text{is} \quad \begin{cases} \text{even if } n \text{ is odd} \\ \text{odd if } n \text{ is even.} \end{cases}$$

transpositions are self-inverse

Note that if

$$\alpha = \tau_1 \tau_2 \dots \tau_{k-1} \tau_k$$

where $\tau_1, \tau_2, \dots, \tau_k$ are transpositions, then

$$\alpha^{-1} = \tau_k^{-1} \tau_{k-1}^{-1} \dots \tau_2^{-1} \tau_1^{-1} = \tau_k \tau_{k-1} \dots \tau_2 \tau_1,$$

so

the parity of α and α^{-1} is the same.

evenness or oddness

Theorem: Every permutation of a finite set is even or odd, but not both.

difficult to prove, see later

Recall

$$\mathrm{Sym}(n) = \{\text{all permutations of } \{1, \dots, n\}\},$$

which is a group under composition, called the **symmetric group** (on n letters).

Put

$$\mathrm{Alt}(n) = \{\text{all even permutations of } \{1,\dots,n\}\},$$

called the <u>alternating group</u> (<u>on n letters</u>),

which is also a group under composition.

$$\boxed{\mathrm{Alt}(n) \text{ is a } \underline{\text{subgroup}} \text{ of } \mathrm{Sym}(n).}$$

closed under composition

- if $\alpha, \beta \in \mathrm{Alt}(n)$ then $\alpha\beta \in \mathrm{Alt}(n)$.

$\boxed{\mathrm{Alt}(n) \text{ is a } \underline{\text{subgroup}} \text{ of } \mathrm{Sym}(n).}$

closed under composition

- if $\alpha, \beta \in \mathrm{Alt}(n)$ then $\alpha\beta \in \mathrm{Alt}(n)$.

$\alpha = \tau_1 \cdots \tau_k$, $\beta = \sigma_1 \cdots \sigma_\ell$

where $\tau_1, \ldots, \tau_k, \sigma_1, \ldots, \sigma_\ell$ are transpositions and k, ℓ are even

$\Longrightarrow$ $\alpha\beta = \tau_1 \cdots \tau_k \sigma_1 \cdots \sigma_\ell$ product of $k+\ell$ transpositions, and $k+\ell$ is also even.

Common to write

$$\boxed{S_n = \mathrm{Sym}(n), \quad A_n = \mathrm{Alt}(n)}.$$

e.g. $S_2 = \{1, (1\,2)\}$,

$A_2 = \{1\}$,

$S_3 = \{1, (1\,2\,3), (1\,3\,2), (1\,2), (1\,3), (2\,3)\}$,

$A_3 = \{1, (1\,2\,3), (1\,3\,2)\}$

$A_4 = \{1, (1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2), (2\,3\,4),$
$(2\,4\,3), (1\,3\,4), (1\,4\,3), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$

Suppose $n \geq 2$.

The single transposition $(1\,2)$ is odd, so

$$(1\,2) \notin A_n .$$

$$\boxed{\underline{\text{Claim}}: \quad S_n = A_n \,\dot{\cup}\, (1\,2)\, A_n}$$

where $\dot{\cup}$ denotes <u>disjoint union</u> (empty intersection)

and

$$(1\,2)\, A_n = \{ (1\,2)\,\varphi \mid \varphi \in A_n \}$$

$$= \{ \text{odd permutations of } \{1, \dots, n\} \}$$

exercise

Suppose $n \geq 2$.

The single transposition $(1\,2)$ is odd, so

$$(1\,2) \notin A_n .$$

coset decomposition

$$\boxed{\underline{\text{Claim}}: \quad S_n = \underbrace{A_n}_{} \,\dot{\cup}\, \underbrace{(1\,2)\,A_n}_{}}$$

even permutations

form a subgroup

odd permutations

form a coset, translation of a subgroup

e.g. $A_3 = \{ 1, (123), (132) \}$

$$(12)A_3 = \{ (12)1, (12)(123), (12)(132) \}$$
$$= \{ (12), (13), (23) \}$$

$$A_4 = \{ 1, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23) \}$$

$$(12)A_4 = \{ (12)1, (12)(123), (12)(132), (12)(124), (12)(142), (12)(134), (12)(143), (12)(234), (12)(243), (12)(12)(34), (12)(13)(24), (12)(14)(23) \}$$

$$= \{ (12), (13), (23), (14), (24), (1234), (1243), (1342), (1432), (34), (1423), (1324) \}$$

Suppose $n \geq 2$.

The single transposition $(1\,2)$ is odd, so

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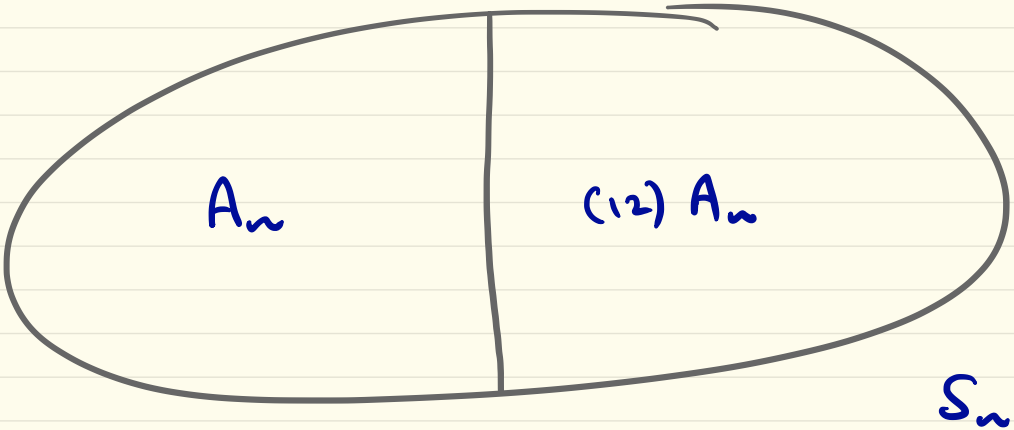
$$\boxed{\underline{\text{Claim}} : \quad S_n = A_n \,\dot{\cup}\, (1\,2)\, A_n}$$

Further

$$\boxed{|A_n| = |(1\,2)\, A_n|} \quad \text{(exercise)}$$

- i.e. the number of even permutations is the same as the number of odd permutations.



$|S_n| = n!$ and $|A_n| = |(12)A_n|$,

so $n! = |S_n| = |A_n| + |(12)A_n|$

$= |A_n| + |A_n| = 2|A_n|$,

so $\boxed{|A_n| = \frac{n!}{2}}$.

$$\boxed{|A_n| = \frac{n!}{2}}$$

e.g.

$$|A_3| = \frac{3!}{2} = \frac{6}{2} = 3,$$

$$|A_4| = \frac{4!}{2} = \frac{24}{2} = 12,$$

$$|A_5| = \frac{5!}{2} = \frac{120}{2} = 60.$$

A_5, the alternating group on 5 letters, is famous in the history of mathematics.

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key example in the proof of the unsolvability of the quintic polynomial equation

Galois theory (first half of 19th century)

Application to the 8-puzzle:

1	2	3
4	5	6
7	8	

allowed to slide squares in and out of the space

Theorem: There are precisely $\frac{8!}{2} = 20{,}160$ possible configurations, where the space is in the lower right corner, corresponding to all <u>even</u> permutations of the square labels.

e.g.

2	3	1
4	5	6
7	8	

corresponds to $(1\ 3\ 2)$, which is even, so possible ✓

3	2	1
4	5	6
7	8	

corresponds to $(1\ 3)$, which is odd, so impossible ✗

e.g.

7	8	3
1	2	6
4	5	

corresponds to $(1\,4\,7)(2\,5\,8)$

which is even, so possible ✓

7	6	1
4	3	8
5	2	

corresponds to $(1\,3\,5\,7)(2\,8\,6)$,

which is odd, so impossible

✗

What about

2	6	3
8		7
4	1	5

?

This is possible iff

2	6	3
8	7	5
4	1	

is possible,

which corresponds to $(1\ 8\ 4\ 7\ 5\ 6\ 2)$,

which is even, so the above configuration is possible.