THE UNIVERSITY OF SYDNEY

MATH2022 LINEAR AND ABSTRACT ALGEBRA

Semester 1 **Week 5 Longer Solutions** 2020

1. (a) Since A is lower triangular, the eigenvalues are 2 and 1. To find the eigenspace corresponding to 2:

$$2I - A = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to 1:

$$I - A = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} 0 \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

(b) Observe that

$$\det(\lambda I - B) = \begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{vmatrix} = (\lambda-1)(\lambda-4)+2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)\ ,$$

yielding eigenvalues 2 and 3. To find the eigenspace corresponding to 2:

$$2I - B = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to 3:

$$3I - B = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

(c) Observe that

$$\det(\lambda I - C) = \begin{vmatrix} \lambda - 1 & -1 \\ 1 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 3) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2\ ,$$

yielding eigenvalue 2 only. To find the eigenspace corresponding to 2:

$$2I - C = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

2. (a) Since A is upper triangular, the eigenvalues are 1, 2 and 3. To find the eigenspace corresponding to 1:

$$I - A = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to 2:

$$2I - A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to 3:

$$3I - A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{yielding} \left\{ \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

(b) Observe that

$$\det(\lambda I - B) = \begin{vmatrix} \lambda & -1 & 0 \\ 2 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2) \begin{vmatrix} \lambda & -1 \\ 2 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda(\lambda - 3) + 2)$$
$$= (\lambda - 2)(\lambda^2 - 3\lambda + 2) = (\lambda - 2)^2(\lambda - 1)\ ,$$

yielding eigenvalues 2 and 1. To find the eigenspace corresponding to 2:

$$2I - B = \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{yielding} \quad \left\{ \begin{bmatrix} \frac{t}{2} \\ t \\ s \end{bmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to 1:

$$I - B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{yielding} \quad \left\{ \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

(c) Observe that

$$\det(\lambda I - C) = \begin{vmatrix} \lambda + 7 & 2 & -6 \\ 2 & \lambda - 1 & -2 \\ 10 & 2 & \lambda - 9 \end{vmatrix} = \begin{vmatrix} \lambda + 7 & 2 & \lambda + 1 \\ 2 & \lambda - 1 & 0 \\ 10 & 2 & \lambda + 1 \end{vmatrix}$$
$$= \begin{vmatrix} \lambda - 3 & 0 & 0 \\ 2 & \lambda - 1 & 0 \\ 10 & 2 & \lambda + 1 \end{vmatrix} = (\lambda - 3)(\lambda - 1)(\lambda + 1)\ ,$$

yielding eigenvalues 3, 1 and -1. To find the eigenspace corresponding to 3:

$$3I - C = \begin{bmatrix} 10 & 2 & -6 \\ 2 & 2 & -2 \\ 10 & 2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -8 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix},$$

yielding $\left\{ \begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$. To find the eigenspace corresponding to 1:

$$I - C = \begin{bmatrix} 8 & 2 & -6 \\ 2 & 0 & -2 \\ 10 & 2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

yielding $\left\{ \begin{bmatrix} t \\ -t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$. To find the eigenspace corresponding to -1:

$$-I - C \; = \; \begin{bmatrix} 6 & 2 & -6 \\ 2 & -2 & -2 \\ 10 & 2 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 8 & 0 \\ 0 & 12 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

yielding $\left\{ \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

3. We have $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector $\mathbf{v}$.

(a) If $A^2 = 0$, the zero matrix, then, denoting the zero column vector by $\mathbf{0}$, we have

$$\mathbf{0} \; = \; A^2\mathbf{v} \; = \; A(A\mathbf{v}) \; = \; A(\lambda\mathbf{v}) \; = \; \lambda(A\mathbf{v}) \; = \; \lambda^2\mathbf{v} \, ,$$

so that $\lambda^2 = 0$, since $\mathbf{v}$ is nonzero, so that $\lambda = 0$.

(b) Suppose that $A^2 = A$. Then

$$\lambda\mathbf{v} \; = \; A\mathbf{v} \; = \; A^2\mathbf{v} \; = \; A(A\mathbf{v}) \; = \; A(\lambda\mathbf{v}) \; = \; \lambda(A\mathbf{v}) \; = \; \lambda^2\mathbf{v} \, .$$

Hence $\mathbf{0} = \lambda^2\mathbf{v} - \lambda\mathbf{v} = (\lambda^2 - \lambda)\mathbf{v} = \lambda(\lambda - 1)\mathbf{v}$, so that $\lambda(\lambda - 1) = 0$, since $\mathbf{v}$ is nonzero. Thus $\lambda = 0$ or $\lambda = 1$.

(c) Suppose that $A^2 = I$. Then

$$\mathbf{v} \; = \; I\mathbf{v} \; = \; A^2\mathbf{v} \; = \; A(A\mathbf{v}) \; = \; A(\lambda\mathbf{v}) \; = \; \lambda(A\mathbf{v}) \; = \; \lambda^2\mathbf{v} \, .$$

Hence $\mathbf{0} = \lambda^2\mathbf{v} - \mathbf{v} = (\lambda^2 - 1)\mathbf{v} = (\lambda - 1)(\lambda + 1)\mathbf{v}$, so that $(\lambda - 1)(\lambda + 1) = 0$, since $\mathbf{v}$ is nonzero. Thus $\lambda = 1$ or $\lambda = -1$.

4. (a) We have $\alpha^6 = \beta^2 = 1$ and $\beta\alpha = \alpha^{-1}\beta = \alpha^5\beta$, from which it follows quickly, by moving all α's to the left and β's to the right and collapsing, that

$$G \; = \{1\, , \; \alpha\, , \; \alpha^2\, , \; \alpha^3\, , \; \alpha^4\, , \; \alpha^5\, , \; \beta\, , \; \alpha\beta\, , \; \alpha^2\beta\, , \; \alpha^3\beta\, , \; \alpha^4\beta\, , \; \alpha^5\beta\} \, .$$

If the order in which the generators are written is reversed then powers of α stay the same (including the identity element 1 vacuously), and β stays the same. However, rewriting $\alpha^i\beta$ backwards, for $1 \le i \le 5$ produces

$$\beta\alpha^i \; = \; \alpha^{-i}\beta \; = \; \alpha^{6-i}\beta$$

so that the list of reflections $\alpha\beta, \alpha^2\beta, \alpha^3\beta, \alpha^4\beta, \alpha^5\beta$ is duplicated but in reverse order.

(b) Using the facts that $\alpha^6 = \beta^2 = 1$ and $\beta\alpha^i\beta = \alpha^{-i}$ for all i, we have

$$\beta\alpha^5\beta^5\alpha^{-3}\beta^{-3}\alpha^8\beta \; = \; \beta\alpha^5\beta\alpha^3\beta\alpha^2\beta \; = \; \beta\alpha^5\alpha^{-3}\alpha^2\beta \; = \; \beta\alpha^4\beta \; = \; \alpha^{-4} \; = \; \alpha^2 \, .$$

(c) Put

$$A = \begin{bmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix},$$

the rotation matrix that rotates 60 degrees anticlockwise, and

$$B = \begin{bmatrix} \cos 0 & \sin 0 \\ \sin 0 & -\cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

the reflection matrix that reflects through the horizontal axis (but in fact any reflection matrix will work). Then $A^6 = B^2 = I$ and

$$BAB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} = A^{-1},$$

so that, as in (a), it follows that

$$H \;=\; \{I\,,\ A\,,\ A^2\,,\ A^3\ ,A^4\,,\ A^5\,,\ B\,,\ AB\,,\ A^2B\,,\ A^3B\,,\ A^4B\,,\ A^5B\}\ .$$

Because all of these powers of A are distinct (corresponding to multiples of one sixth rotation), it follows that all elements of H are distinct. The mapping $\alpha \mapsto A$ and $\beta \mapsto B$ induces a bijection from G to H that respects the group multiplication since, on the one hand, for $0 \le i, j \le 5$ and $0 \le k, \ell \le 1$,

$$(\alpha^i\beta^k)(\alpha^j\beta^\ell) \;=\; \begin{cases} \alpha^{i+j}\beta^\ell & \text{if } k = 0, \\ \alpha^{i-j}\beta^{1+\ell} & \text{if } k = 1, \end{cases}$$

whilst, on the other hand,

$$(A^iB^k)(A^jB^\ell) \;=\; \begin{cases} A^{i+j}B^\ell & \text{if } k = 0, \\ A^{i-j}B^{1+\ell} & \text{if } k = 1, \end{cases}$$

where addition or subtraction of exponents is taken mod 6 and mod 2.

5. We have $\alpha^6 = \beta^2 = 1$, since α is a single 6-cycle and β is a product of disjoint transpositions (2-cycles). Further $\beta^{-1} = \beta$ and

$$\beta^{-1}\alpha\beta = \beta\alpha\beta = (1\ 6)(2\ 5)(3\ 4)(1\ 2\ 3\ 4\ 5\ 6)(1\ 6)(2\ 5)(3\ 4) = (1\ 6\ 5\ 4\ 3\ 2) = \alpha^{-1}\ .$$

Thus

$$\beta\alpha \;=\; \beta\alpha\beta\beta \;=\; \alpha^{-1}\beta \;=\; \alpha^5\beta\ .$$

Collecting all the α's to the left and β's to the right, and simplifying, it follows that

$$\begin{aligned} \langle\alpha,\beta\rangle &= \{1\,,\ \alpha\,,\ \alpha^2\,,\ \alpha^3\,,\ \alpha^4\,,\ \alpha^5\,,\ \beta\,,\ \alpha\beta\,,\ \alpha^2\beta\,,\ \alpha^3\beta\,,\ \alpha^4\beta\,,\ \alpha^5\beta\} \\ &= \{1, (1\ 2\ 3\ 4\ 5\ 6), (1\ 3\ 5)(2\ 4\ 6), (1\ 4)(2\ 5)(3\ 6), (1\ 5\ 3)(2\ 6\ 4), (1\ 6\ 5\ 4\ 3\ 2), \\ &\quad (1\ 6)(2\ 5)(3\ 4), (1\ 5)(2\ 4), (1\ 4)(2\ 3)(5\ 6), (1\ 3)(4\ 6), (1\ 2)(3\ 6)(4\ 5), (2\ 6)(3\ 5)\}. \end{aligned}$$

We want $\gamma \in \mathrm{Sym}(6)$ such that $\gamma^{-1}\alpha\gamma = \alpha^{-1}$, so we look exhaustively for permutations that rewrite the cycle $(1\ 2\ 3\ 4\ 5\ 6)$ as all possible cycles that represent α^{-1}, namely,

$$(1\ 6\ 5\ 4\ 3\ 2)\,,\ (6\ 5\ 4\ 3\ 2\ 1)\,,\ (5\ 4\ 3\ 2\ 1\ 6)\,,\ (4\ 3\ 2\ 1\ 6\ 5)\,,\ (3\ 2\ 1\ 6\ 5\ 4)\,,\ (2\ 1\ 6\ 5\ 4\ 3)\,,$$

producing the following possibilities, in the same order:

$$\gamma = (2\,6)(3\,5) = \alpha^5\beta\ , \quad \gamma = (1\,6)(2\,5)(3\,4) = \beta\ , \quad \gamma = (1\,5)(2\,4) = \alpha\beta\ ,$$

$$\gamma = (1\,4)(2\,3)(5\,6) = \alpha^2\beta\ , \quad \gamma = (1\,3)(4\,6) = \alpha^3\beta\ , \quad \gamma = (1\,2)(3\,6)(4\,5) = \alpha^4\beta\ .$$

These possibilities exhaust precisely all of the reflections of the form $\alpha^i\beta$ for $0 \le i \le 5$.

6. (a) We have that

$$M^2 = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 12 \\ 6 & 13 & 24 \\ -3 & -6 & -11 \end{bmatrix} ,$$

and

$$3M - 2I = \begin{bmatrix} 6 & 6 & 12 \\ 6 & 15 & 24 \\ -3 & -6 & -9 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 12 \\ 6 & 13 & 24 \\ -3 & -6 & -11 \end{bmatrix} ,$$

which verifies that $M^2 = 3M - 2I$. Further

$$\begin{aligned} \chi(\lambda) &= \det(\lambda I - M) = \begin{vmatrix} \lambda-2 & -2 & -4 \\ -2 & \lambda-5 & -8 \\ 1 & 2 & \lambda+3 \end{vmatrix} = \begin{vmatrix} \lambda-2 & -2 & -4 \\ -2 & \lambda-5 & -8 \\ \lambda-1 & 0 & \lambda-1 \end{vmatrix} \\ &= \begin{vmatrix} \lambda+2 & -2 & -4 \\ 6 & \lambda-5 & -8 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda+2 & -2 \\ 6 & \lambda-5 \end{vmatrix} \\ &= (\lambda-1)\big((\lambda+2)(\lambda-5)+12\big) = (\lambda-1)(\lambda^2-3\lambda+2) = (\lambda-1)^2(\lambda-2)\ . \end{aligned}$$

(b) By the Cayley-Hamilton Theorem,

$$\chi(M) = (M-I)^2(M-2I) = 0\ .$$

This is consistent with the first part of (a), since $M^2 = 3M - 2I$ implies that

$$(M-I)(M-2I) = M^2 - 3M + 2I = 0\ ,$$

which in turn implies $\chi(M) = 0$, since $(\lambda-1)(\lambda-2)$ divides $\chi(\lambda)$.

(c) The formula holds trivially for $k = 1$, and also for $k = 2$ by part (a), which starts an induction. Suppose $k > 2$ and, as inductive hypothesis, that

$$M^{k-1} = (2^{k-1}-1)M + (2-2^{k-1})I\ .$$

Then

$$\begin{aligned} M^k &= MM^{k-1} = M\big((2^{k-1}-1)M + (2-2^{k-1})I\big) = (2^{k-1}-1)M^2 + (2-2^{k-1})M \\ &= (2^{k-1}-1)(3M-2I) + (2-2^{k-1})M = (3(2^{k-1}-1)+2-2^{k-1})M + (2-2^k)I \\ &= (2^k-1)M + (2-2^k)I\ , \end{aligned}$$

which verifies the inductive step, completing the proof for all positive k. The formula also holds trivially for $k = 0$. One can prove the formula for negative k also by induction. A direct verification is to calculate as follows, for positive k:

$$\begin{aligned}
&M^k\Big((2^{-k}-1)M + (2-2^{-k})I\Big) \\
&\quad = \Big((2^k-1)M + (2-2^k)I\Big)\Big((2^{-k}-1)M + (2-2^{-k})I\Big) \\
&\quad = \Big((2^k-1)(2^{-k}-1\Big)M^2 + \Big((2^k-1)(2-2^{-k}) + (2-2^k)(2^{-k}-1)\Big)M \\
&\qquad\qquad + \big(2-2^k)(2-2^{-k}\big)I \\
&\quad = \big(2-2^k-2^{-k}\big)M^2 + \big(2^{k+1}-3+2^{-k}+2^{-k+1}-3+2^k\big)M \\
&\qquad\qquad + \big(5-2^{k+1}-2^{-k+1}\big)I \\
&\quad = \big(2-2^k-2^{-k}\big)\big(3M-2I\big) + \big(3(2^k)-6+3(2^{-k})\big)M + \big(5-2^{k+1}-2^{-k+1}\big)I \\
&\quad = \Big(6-3(2^k)-3(2^{-k})+3(2^k)-6+3(2^{-k})\Big)M \\
&\qquad\qquad + \Big(-4+2^{k+1}+2^{-k+1}+5-2^{k+1}-2^{-k+1}\Big)I \\
&\quad = I\ ,
\end{aligned}$$

which verifies that

$$M^{-k} = (M^k)^{-1} = (2^{-k}-1)M + (2-2^{-k})I\ ,$$

so that the formula holds for all integers k.

(d) The formula gives

$$\begin{aligned}
M^5 &= (2^5-1)M + (2-2^5)I = 31\begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -3 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{bmatrix} \\
&= \begin{bmatrix} 32 & 62 & 124 \\ 62 & 125 & 248 \\ -31 & -62 & -123 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
M^{-1} &= (2^{-1}-1)M + (2-2^{-1})I = -\frac{1}{2}\begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -3 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & -1 & -2 \\ -1 & -1 & -4 \\ \frac{1}{2} & 1 & 3 \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
M^{-5} &= (2^{-5}-1)M + (2-2^{-5})I = -\frac{31}{32}\begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -3 \end{bmatrix} + \begin{bmatrix} \frac{63}{32} & 0 & 0 \\ 0 & \frac{63}{32} & 0 \\ 0 & 0 & \frac{63}{32} \end{bmatrix} \\
&= \frac{1}{32}\begin{bmatrix} 1 & -62 & -124 \\ -62 & -92 & -248 \\ 31 & 62 & 156 \end{bmatrix}.
\end{aligned}$$

7. (a) We have

$$A^2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix},$$

after simplifying, which corresponds to a rotation of the plane 2θ radians.

(b) We have

$$B^2 = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I\,,$$

after simplifying, which corresponds to a rotation of the plane 0 radians (the identity mapping).

(c) From general facts about rotation matrices (see Q4(a)(b) of Week 2 Exercises),

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^3 = \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix},$$

which corresponds to a rotation of the plane -3θ radians.

(d) We have

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} = \begin{bmatrix} \cos(\theta+2\phi) & \sin(\theta+2\phi) \\ \sin(\theta+2\phi) & -\cos(\theta+2\phi) \end{bmatrix},$$

after simplifying, which corresponds to a reflection of the plane through the line through the origin making an angle of $\frac{\theta+2\phi}{2}$ with the positive x-axis.

(e) As a special case of the previous calculation, taking $\phi=\theta$, we have that

$$AC = \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ \sin 3\theta & -\cos 3\theta \end{bmatrix},$$

which corresponds to a reflection of the plane through the line through the origin making an angle of $\frac{3\theta}{2}$ with the positive x-axis.

(f) We have

$$BA = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(2\phi-\theta) & \sin(2\phi-\theta) \\ \sin(2\phi-\theta) & -\cos(2\phi-\theta) \end{bmatrix},$$

after simplifying, which corresponds to a reflection of the plane through the line through the origin making an angle of $\frac{2\phi-\theta}{2}$ with the positive x-axis.

(g) We have

$$BC = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos 2(\phi-\theta) & -\sin 2(\phi-\theta) \\ \sin 2(\phi-\theta) & \cos 2(\phi-\theta) \end{bmatrix},$$

after simplifying, which corresponds to a rotation of the plane $2(\phi-\theta)$ radians.

(h) We have, by (d) and (f),

$$\begin{aligned} ABA &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta+2\phi) & \sin(\theta+2\phi) \\ \sin(\theta+2\phi) & -\cos(\theta+2\phi) \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta+2\phi-\theta) & \sin(\theta+2\phi-\theta) \\ \sin(\theta+2\phi-\theta) & -\cos(\theta+2\phi-\theta) \end{bmatrix} = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} = B. \end{aligned}$$

Alternatively, by Q4(a)(d) of Week 2 Exercises,

$$ABA \;=\; ABAI \;=\; ABABB \;=\; A(BAB)B \;=\; AA^{-1}B \;=\; IB \;=\; B\,.$$

(i) By Q4(a)(d) of Week 2 Exercises,

$$BA^2B \;=\; (A^2)^{-1} \;=\; A^{-2}\,,$$

which corresponds to a rotation of the plane -2θ radians.

(j) We have, by (f) and (g),

$$\begin{aligned} BAC &= \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\phi-\theta) & \sin(2\phi-\theta) \\ \sin(2\phi-\theta) & -\cos(2\phi-\theta) \end{bmatrix} \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\phi-3\theta) & -\sin(2\phi-3\theta) \\ \sin(2\phi-3\theta) & \cos(2\phi-3\theta) \end{bmatrix}, \end{aligned}$$

which corresponds to a rotation of the plane $2\phi - 3\theta$ radians.

8. We have

$$\chi(\lambda) = \det(\lambda I - M) = \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix} = (\lambda-a)(\lambda-d) - bc = \lambda^2 - (a+d)\lambda + ad - bc\,,$$

and

$$\begin{aligned} \chi(M) &= M^2 - (a+d)M + (ad-bc)I \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+db \\ ac+dc & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} a^2+bc-a^2-ad+ad-bc & ab+bd-ab-db \\ ca+dc-ac-dc & cb+d^2-ad-d^2+ad-bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

the zero matrix, verifying of the Cayley-Hamilton Theorem for 2×2 matrices.

9. (a) We have that $M\mathbf{v} = \lambda\mathbf{v}$ for some nonzero column vector $\mathbf{v}$. We claim that $M^k\mathbf{v} = \lambda^k\mathbf{v}$ for all positive integers k, and verify this by induction. This is clearly true if $k = 1$. Suppose that $k \geq 1$ and assume as inductive hypothesis that $M^k\mathbf{v} = \lambda^k\mathbf{v}$. Then

$$M^{k+1}\mathbf{v} = (MM^k)\mathbf{v} = M(M^k\mathbf{v}) = M(\lambda^k\mathbf{v}) = \lambda^k(M\mathbf{v}) = \lambda^k(\lambda\mathbf{v}) = \lambda^{k+1}\mathbf{v}\ ,$$

which establishes the inductive step, and completes the proof of our claim. Thus, for all positive integers k, we have that λ^k is an eigenvalue of M^k (and further that $\mathbf{v}$ is always a corresponding eigenvector).

(b) Again, we have that $M\mathbf{v} = \lambda\mathbf{v}$ for some nonzero column vector $\mathbf{v}$. If $\lambda = 0$ then $M\mathbf{v} = 0\mathbf{v} = \mathbf{0}$, so that

$$\mathbf{v} \;=\; I\mathbf{v} \;=\; (M^{-1}M)\mathbf{v} \;=\; M^{-1}(M\mathbf{v}) \;=\; M^{-1}\mathbf{0} \;=\; \mathbf{0}\ ,$$

which contradicts that $\mathbf{v}$ is a nonzero column vector. Hence λ is nonzero. Further,

$$\mathbf{v} \;=\; \frac{1}{\lambda}(\lambda\mathbf{v}) \;=\; \lambda^{-1}M\mathbf{v}\ ,$$

so that

$$M^{-1}\mathbf{v} \;=\; M^{-1}\big(\lambda^{-1}M\mathbf{v}\big) \;=\; \lambda^{-1}(M^{-1}M)\mathbf{v} \;=\; \lambda^{-1}I\mathbf{v} \;=\; \lambda^{-1}\mathbf{v}\ .$$

This proves that that λ^{-1} is an eigenvalue of M^{-1} (and further that $\mathbf{v}$ is a corresponding eigenvector).

10. Let $M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where $\theta \in \mathbb{R}$ be a rotation matrix. Working over $\mathbb{C}$, we have

$$\begin{aligned}\chi(\lambda) \;&=\; \det(\lambda I - M) \;=\; \begin{vmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{vmatrix} \;=\; (\lambda - \cos\theta)^2 + \sin^2\theta \\ &=\; \lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta \;=\; \lambda^2 - 2\lambda\cos\theta + 1\ ,\end{aligned}$$

with roots

$$\lambda \;=\; \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \;=\; \cos\theta \pm i\sin\theta \;=\; e^{\pm i\theta}$$

where $i = \sqrt{-1}$. If θ is an even multiple of π then $M = I$ and $\lambda = 1$ is the unique eigenvalue. If θ is an odd multiple of π then $M = -I$ and $\lambda = -1$ is the unique eigenvalue. In both these cases, the eigenspace comprises all column vectors with entries from $\mathbb{C}$. Suppose then that θ is not a multiple of π, so that $\sin\theta \neq 0$. Finding the eigenspace corresponding to $\lambda = e^{i\theta} = \cos\theta + i\sin\theta$:

$$\begin{bmatrix} i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{bmatrix} \sim \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}\ ,$$

yielding $\left\{ \begin{bmatrix} iz \\ z \end{bmatrix} \;\middle|\; z \in \mathbb{C} \right\}$. Finding the eigenspace corresponding to $\lambda = e^{-i\theta} = \cos\theta - i\sin\theta$:

$$\begin{bmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{bmatrix} \sim \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}\ ,$$

yielding $\left\{ \begin{bmatrix} -iz \\ z \end{bmatrix} \;\middle|\; z \in \mathbb{C} \right\}$.

11. The (i,ℓ)-entry of $A(\mathrm{adj}A)$ is

$$\sum_{j=1}^{n}(-1)^{\ell+j}a_{ij}\det A_{\ell j}\ .$$

If $i=\ell$ then this is the expansion along the ℓth row of A, yielding $\det A$. If $i\neq\ell$ then this is the expansion along the ℓth row of the matrix obtained by replacing the ℓth row of A by the ith row, yielding the determinant of a matrix with two identical rows, which is zero. This proves $A(\mathrm{adj}A)=(\det A)I$, and then the rest follows quickly.

12. Given $M\mathbf{x}=\mathbf{c}$, we have

$$\mathbf{x} \;=\; M^{-1}\mathbf{c} \;=\; \frac{1}{\det M}(\mathrm{adj}M)\mathbf{c}\ ,$$

so that, for each i,

$$x_i \;=\; \frac{1}{\det M}\sum_{j=1}^{n}(-1)^{j+i}\det(M_{ji})c_j \;=\; \frac{\det M_i}{\det M}\ ,$$

since $\det M_i=\sum_{j=1}^{n}(-1)^{j+i}c_j\det M_{ji}$, expanding down the ith column. In the given system,

$$M \;=\; \begin{bmatrix} 2 & 3 & 4\\ 5 & 5 & 6\\ 3 & 1 & 2\end{bmatrix} \quad\text{and}\quad \mathbf{c} \;=\; \begin{bmatrix} -4\\ -3\\ -1\end{bmatrix}\ ,$$

so that

$$\det M \;=\; \begin{vmatrix} 2 & 3 & 4\\ 5 & 5 & 6\\ 3 & 1 & 2\end{vmatrix} \;=\; 2\begin{vmatrix} 5 & 6\\ 1 & 2\end{vmatrix} - 3\begin{vmatrix} 5 & 6\\ 3 & 2\end{vmatrix} + 4\begin{vmatrix} 5 & 5\\ 3 & 1\end{vmatrix} \;=\; -8\ ,$$

$$\det M_1 \;=\; \begin{vmatrix} -4 & 3 & 4\\ -3 & 5 & 6\\ -1 & 1 & 2\end{vmatrix} \;=\; -4\begin{vmatrix} 5 & 6\\ 1 & 2\end{vmatrix} - 3\begin{vmatrix} -3 & 6\\ -1 & 2\end{vmatrix} + 4\begin{vmatrix} -3 & 5\\ -1 & 1\end{vmatrix} \;=\; -8\ ,$$

$$\det M_2 \;=\; \begin{vmatrix} 2 & -4 & 4\\ 5 & -3 & 6\\ 3 & -1 & 2\end{vmatrix} \;=\; 2\begin{vmatrix} -3 & 6\\ -1 & 2\end{vmatrix} + 4\begin{vmatrix} 5 & 6\\ 3 & 2\end{vmatrix} + 4\begin{vmatrix} 5 & -3\\ 3 & -1\end{vmatrix} \;=\; -16\ ,$$

$$\det M_3 \;=\; \begin{vmatrix} 2 & 3 & -4\\ 5 & 5 & -3\\ 3 & 1 & -1\end{vmatrix} \;=\; 2\begin{vmatrix} 5 & -3\\ 1 & -1\end{vmatrix} - 3\begin{vmatrix} 5 & -3\\ 3 & -1\end{vmatrix} - 4\begin{vmatrix} 5 & 5\\ 3 & 1\end{vmatrix} \;=\; 24\ ,$$

yielding

$$x \;=\; \frac{-8}{-8} \;=\; 1\ ,\quad y \;=\; \frac{-16}{-8} \;=\; 2\ ,\quad z \;=\; \frac{24}{-8} \;=\; -3\ .$$