The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Week 6 Hints and Short Solutions** 2020

1. (a) $\begin{bmatrix} 2^k & 0 \\ 2^k-1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2^{k+1}-3^k & -2^k+3^k \\ 2^{k+1}-2(3^k) & -2^k+2(3^k) \end{bmatrix}$ (c) $\frac{1}{2}\begin{bmatrix} 1+(-1)^k & -1+(-1)^k \\ -1+(-1)^k & 1+(-1)^k \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2^k-1 & 2^k-1 \\ 0 & 2^k & 2^k-3^k \\ 0 & 0 & 3^k \end{bmatrix}$ (e) $\begin{bmatrix} 2-2^k & 2^k-1 & 0 \\ 2-2^{k+1} & 2^{k+1}-1 & 0 \\ 0 & 0 & 2^k \end{bmatrix}$

(f) $\begin{bmatrix} 3(-1)^k-3^k-1 & (-1)^k-1 & 3^k+1-2(-1)^k \\ 1-3^k & 1 & 3^k-1 \\ 3(-1)^k-2(3^k)-1 & (-1)^k-1 & 2(3^k)+1-2(-1)^k \end{bmatrix}$

2. $\begin{bmatrix} \frac{4}{9} \\ \frac{5}{9} \end{bmatrix}$, $M^k = \begin{bmatrix} \frac{4}{9}+\frac{5}{9}10^{-k} & \frac{4}{9}-\frac{4}{9}10^{-k} \\ \frac{5}{9}-\frac{5}{9}10^{-k} & \frac{5}{9}+\frac{4}{9}10^{-k} \end{bmatrix} \to \begin{bmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{5}{9} & \frac{5}{9} \end{bmatrix}$ as $k \to \infty$.

3. (a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 13 \\ 18 \end{bmatrix}, \begin{bmatrix} 80 \\ 111 \end{bmatrix}, \begin{bmatrix} 493 \\ 684 \end{bmatrix}, \begin{bmatrix} 3038 \\ 4215 \end{bmatrix}$.

(b) $\lambda^2 - 6\lambda - 1$ with roots $3+\sqrt{10} \approx 6.16228$ and $3-\sqrt{10} \approx -0.16228$.

(c) Check that $A\mathbf{v} \approx (3+\sqrt{10})\mathbf{v}$ where $\mathbf{v} = \frac{1}{3038}\mathbf{v}_5$.

4. (a) $\begin{bmatrix} -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 25 \\ -18 \end{bmatrix}, \begin{bmatrix} -154 \\ 111 \end{bmatrix}, \begin{bmatrix} 949 \\ -684 \end{bmatrix}, \begin{bmatrix} -5848 \\ 4215 \end{bmatrix}$.

(b) Check that $B\mathbf{w} \approx -6.16228\,\mathbf{w}$.

(c) The eigenvalues of B are the reciprocals of the eigenvalues of A.

5. $x_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

6. (a) Columns of M add up to 1 and entries of M^2 are all positive. (b) $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

7. Scalar matrices commute with all matrices.

8. Roots of a real polynomial come in complex conjugate pairs.

9. Both matrices have exactly one eigenvalue.

10. For one direction multiply through by the inverse matrix. For the other direction use the fact that a square matrix is invertible if and only if it row reduces to the identity matrix in reduced row echelon form.

11. Observe that λI is both upper and lower triangular.

12. For part (a), use commutativity of multiplication in the underlying field and change the order of summation in a double summation.