

THE UNIVERSITY OF SYDNEY
MATH2022 LINEAR AND ABSTRACT ALGEBRA

Semester 1

Week 7 Hints and Short Solutions

2020

1. (a) $\{(0, 0), (0, 1), (1, 0), (1, 1)\}, \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}, \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}, \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$
 (c) Try $(1, 1)$ or $(1, 2)$.
 (d) Try $(1, 0), (0, 1)$. One ordered pair generates too few elements.
 (e) Try $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. Two ordered triples generate too few elements.
2. For converse, observe that $\mathbf{v} + \mathbf{w} = 1\mathbf{v} + 1\mathbf{w}$ and $\lambda\mathbf{v} = \lambda\mathbf{v} + 0\mathbf{w}$ are linear combinations.
3. $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
4. (b) $M_f = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 3 & -2 \end{bmatrix}, M_g = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
 (c) $gf(x, y) = (3x + 2y, -x + 4y, -2x - 3y, 5x - y), M_{gf} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \\ -2 & -3 \\ 5 & -1 \end{bmatrix}$
5. (a) rotation $\tan^{-1} 0.5$ clockwise; dilation by $\frac{1}{\sqrt{5}}$ and by $\frac{\sqrt{5}}{3}$; shear left by $\frac{4}{5}$.
 (b) $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \frac{3}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{bmatrix}$ (c) $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \frac{3}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{bmatrix}$
6. Use facts, in succession, that L_1 and L_2 preserve linear combinations.
7. (a) $M_f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, M_g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, M_h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
 (b) $(x, y) \mapsto (-x - 2y, 2x + y), (-2x - y, -x - 2y), (x + 2y, -2x - y)$
 (d) $(x, y) \mapsto \frac{1}{3}(2x - y, -x + 2y), (y, -x), (y, x), \frac{1}{3}(x + 2y, -2x - y)$
 (e) quarter rotation, reflection in $y = x$.
8. (a) $\emptyset, |A||B|$ (b) $((a, b), c) \mapsto (a, b, c)$ (c) $(a, b) \mapsto (b, a)$ (d) $\{(a_1, a_2, \dots) \mid \text{each } a_i \in \mathbb{Z}_2\}$
9. Use respective group properties holding in each coordinate.
10. (a) Induced by $1 \mapsto (1, 1)$ or $1 \mapsto (1, 2)$.
 (b) Try $(1, 1, 1)$. Any single element of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$ generates at most 10 elements.
11. For composites, mimic the proof for linear transformations.
12. $1, (1\ 2\ 3)(4\ 5), (1\ 3\ 2), (4\ 5), (1\ 2\ 3), (1\ 3\ 2)(4\ 5), (1\ 3), (1\ 2)(4\ 5), (2\ 3), (1\ 3)(4\ 5), (1\ 2), (2\ 3)(4\ 5)$