

Important Ideas and Useful Facts:

- (i) **Linear dependence and independence:** Let V be a vector space over a field F , and $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ for some $k \geq 1$. We call the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ and the set $X = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *linearly independent* if, for all $\lambda_1, \dots, \lambda_k \in F$,

$$\lambda_1 \mathbf{v}_1 + \dots + \lambda_k \mathbf{v}_k = \mathbf{0} \quad \text{implies} \quad \lambda_1 = \dots = \lambda_k = 0,$$

equivalently, in the case $k > 1$, no vector from X can be expressed as a linear combination of other vectors from X . We say that they are *linearly dependent* otherwise, that is, if $X = \{\mathbf{0}\}$ or at least one vector from X can be expressed as a linear combination of other vectors from X . In particular if $\mathbf{0} \in X$, then X is linearly dependent. If $k = 1$ then X is linearly independent if and only if $\mathbf{v}_1$ is nonzero. If $k = 2$ then X is linearly independent if and only if neither of $\mathbf{v}_1$ nor $\mathbf{v}_2$ is a scalar multiple of the other. The empty set $\emptyset$ is declared by definition to be *linearly independent*. If Y is an infinite subset of V then we say that Y is *linearly independent* if every finite subset is linearly independent, and otherwise *linearly dependent*.

- (ii) **Basis and dimension of a vector space:** A *basis* for a vector space V is a linearly independent subset B that spans V . In particular, the empty set is a basis for the trivial vector space. It follows, when B is nonempty, that every vector in V can be expressed uniquely (up to the order of the vectors) as a linear combination of elements of B . In applications, a basis B is typically a nonempty finite ordered list of vectors (and order is important with respect to building matrices, see later). It is an important theorem that every vector space V has a basis and every basis for V has the same size (even when the size is infinite). The size of any basis for V is called the *dimension* of the vector space and denoted by $\dim(V)$. It is another important theorem that every linearly independent subset can be extended to a basis, and every spanning set contains a basis. It follows that, if V is known to be finite dimensional of dimension n , then any linearly independent set or any spanning set of size n is automatically a basis for V .

- (iii) **Standard bases:** Let F be any field. If $n \geq 1$ then the *standard basis* for F^n is

$$B = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$$

where $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ with i in the i th place, for $i = 1, \dots, n$. In particular, F^n has dimension n . The empty set $\emptyset$ is the basis for any trivial vector space (such as F^0), so the dimension of any trivial vector space is zero. Let $\mathbb{P}_n$ denote the vector space of polynomials in x over F of degree at most n , where $n \geq 0$. Then the *standard basis* for $\mathbb{P}_n$ is

$$B = \{1, x, \dots, x^n\}.$$

In particular, $\mathbb{P}_n$ has dimension $n + 1$.

- (iv) **Coordinates of a vector with respect to a basis:** Let V be a vector space over a field F of dimension n and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be an ordered basis for V . Let $\mathbf{v} \in V$. Then there are unique scalars $\lambda_1, \dots, \lambda_n \in F$ such that

$$\mathbf{v} = \lambda_1 \mathbf{b}_1 + \dots + \lambda_n \mathbf{b}_n .$$

We define the *coordinate vector (coordinates) of $\mathbf{v}$ with respect to B* to be the following column vector:

$$[\mathbf{v}]_B = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

If $V = F^n$ and B is the standard basis for V then $[\mathbf{v}]_B = \mathbf{v}^\top$, for all $\mathbf{v} \in V$.

- (v) **Vector spaces with the same dimension are isomorphic:** If V is a vector space over a field F having a basis B with $n \geq 1$ elements, so has dimension n , then V is isomorphic to F^n under the mapping $\mathbf{v} \mapsto [\mathbf{v}]_B^\top$ (for $\mathbf{v} \in V$), where the row vector $[\mathbf{v}]_B^\top$ is, as usual, identified with the n -tuple in F^n . Obviously, all trivial vector spaces, that is, vector spaces of dimension zero, are isomorphic to F^0 .
- (vi) **Isomorphic vector spaces have the same dimension:** If V and W are isomorphic vector spaces over a field F and B is a basis for V , then it follows that the image of B under the isomorphism is a basis for W , and so V and W have the same dimension.
- (vii) **Nonzero rows of a matrix in row echelon form are linearly independent:** The nonzero rows of a matrix M (over a field F) in row echelon form are linearly independent and therefore form a basis for the row space of any matrix over F that can be row reduced to yield the same nonzero rows as M .
- (viii) **Rank of a matrix:** It is an important theorem that the row and column spaces of a matrix M have the same dimension, called the *rank* of M , denoted by $\text{rank}(M)$. The rank is the number of nonzero rows when M or $M^\top$ is row reduced to row echelon form.
- (ix) **Nullity of a matrix:** Let M be an $m \times n$ matrix over a field F . Recall that the *null space* of M may refer either to the vector space

$$\{\text{column vectors } \mathbf{v} \text{ of length } n \mid M\mathbf{v} = \mathbf{0}\} ,$$

or the solution space of the associated homogeneous system of m equations in n variables:

$$\{\mathbf{x} \in F^n \mid M\mathbf{x}^\top = \mathbf{0}\} .$$

The dimension of the null space is called the *nullity* of M , denoted by $\text{nullity}(M)$. The nullity of M is the number of parameters that need to be introduced to yield the solution of the associated homogeneous system of equations.

- (x) **Rank-Nullity Theorem for matrices:** If M is an $m \times n$ matrix then $\text{rank}(M) + \text{nullity}(M) = n$.

Tutorial Exercises:

1. Explain why $\{1, i\}$ is a basis for $\mathbb{C}$ as a vector space over $\mathbb{R}$ where $i = \sqrt{-1}$ (so that $\mathbb{C}$ becomes two dimensional).
2. Explain why $B = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$ is basis for $\mathbb{R}^3$ and find the coordinates of $\mathbf{v}$ with respect to B in the following cases:

(a) $\mathbf{v} = (3, 1, -4)$

(b) $\mathbf{v} = (1, 0, 0)$

(c) $\mathbf{v} = (2, 1, 0)$

3. Consider the following real matrices:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & -4 \\ 6 & 5 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 8 & -11 \\ 16 & 10 & 9 \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 1 & 3 & -4 & 6 & 5 & 4 \\ 3 & 8 & -11 & 16 & 10 & 9 \end{bmatrix}.$$

Row reduce M and $M^\top$ and observe that they have the same rank. Explain why A , B and C are linearly dependent. Express one of A , B , C as a linear combination of the other two.

4. Find a basis for the row space and a basis for the column space of the following real matrix:

$$M = \begin{bmatrix} 2 & 1 & 0 & -4 \\ 3 & 0 & -1 & 2 \\ 6 & -3 & -4 & 20 \end{bmatrix}$$

Verify that the row space and column space of M have the same dimension. Now find a basis for the null space of M . Verify that the Rank-Nullity Theorem holds in this case.

5. Decide whether the following sets of vectors from $\mathbb{R}^{\mathbb{R}}$ (denoted by the rule for their outputs given inputs $x \in \mathbb{R}$) are linearly independent:

(a) $\{1 + x + x^2, 1 - x, 2 + x^2\}$

(b) $\{1 - x - x^2, 1 + x^2, 1 + x + x^2 + x^3, 1 - x^3\}$

(c) $\{\sin x, \cos x\}$

(d) $\{1, \cos 2x, \sin^2 x\}$

- 6.* Recall that $\mathbb{Q}$ is the field of rational numbers and that $\sqrt{2} \notin \mathbb{Q}$. Put

$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}.$$

Prove that $\mathbb{Q}(\sqrt{2})$ is closed under addition and multiplication and taking inverses of nonzero elements. It follows that $\mathbb{Q}(\sqrt{2})$ is a field, and becomes a vector space over $\mathbb{Q}$ by restricting scalar multiplication. Explain why $\{1, \sqrt{2}\}$ is a basis for $\mathbb{Q}(\sqrt{2})$ (so that $\mathbb{Q}(\sqrt{2})$ becomes two dimensional as a vector space over $\mathbb{Q}$).

Further Exercises:

7. Explain why $B = \{1, x - 1, (x - 1)^2\}$ is a basis for the vector space $\mathbb{P}_2$ of real polynomials of degree at most 2. Find the coordinates of $p(x)$ with respect to B in the following cases:
- (a) $p(x) = 2x^2 - 5x + 6$ (b) $p(x) = x^2 + 1$ (c) $p(x) = x^2 - 1$
8. Let F be any field. Find a basis for $\text{Mat}_{2,3}$, the set of 2×3 matrices over F , regarded as a vector space over F with respect to usual matrix addition and scalar multiplication. More generally, explain why $\text{Mat}_{m,n}$ becomes an mn -dimensional vector space over F , for any $m, n \geq 1$.
9. Find the rank and nullity of the following matrices, and a basis for the null space in each case:
- (a) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ over $\mathbb{R}$, $\mathbb{Z}_2$ and $\mathbb{Z}_3$. (b) $B = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$ over $\mathbb{R}$ and $\mathbb{Z}_5$.
- (c) $C = \begin{bmatrix} -1 & 0 & 3 & -2 \\ -1 & 1 & 0 & 3 \\ -1 & 0 & -2 & 3 \end{bmatrix}$ over $\mathbb{R}$ and $\mathbb{Z}_5$.
10. Use the previous exercise, or otherwise, to decide which of the following sets of vectors are linearly independent, as subsets of F^n for appropriate F and n :
- (a) $X = \{(0, 1, 1), (1, 0, 1), (0, 0, 1)\}$ over $\mathbb{R}$, $\mathbb{Z}_2$ and $\mathbb{Z}_3$.
(b) $X = \{(1, -1, -1), (0, 3, 4), (1, 0, 2)\}$ over $\mathbb{R}$ and $\mathbb{Z}_5$.
(c) $X = \{(1, 0, 1), (-1, 3, 0), (-1, 4, 2)\}$ over $\mathbb{R}$ and $\mathbb{Z}_5$.
(d) $X = \{(-1, 0, 3, -2), (-1, 1, 0, 3), (-1, 0, -2, 3)\}$ over $\mathbb{R}$ and $\mathbb{Z}_5$.
(e) $X = \{(-1, -1, -1), (0, 1, 0), (3, 0, -2), (-2, 3, 3)\}$ over $\mathbb{R}$ and $\mathbb{Z}_5$.
11. Verify carefully, from the definition, that if $\mathbf{v}$ and $\mathbf{w}$ are vectors from a vector space V over a field F then $\mathbf{v}$ and $\mathbf{w}$ are linearly independent if and only if neither $\mathbf{v}$ nor $\mathbf{w}$ can be expressed as a scalar multiple of the other.
12. Suppose that $k > 1$ and $\mathbf{v}_1, \dots, \mathbf{v}_k$ are vectors from a vector space. Verify carefully from the definition that $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent if and only if no vector from this list can be expressed as a linear combination of other vectors from the list.
- 13.* Prove carefully that isomorphic vector spaces over the same field have the same dimension.
- 14.* Let $\mathbf{v}$ and $\mathbf{w}$ be eigenvectors for a square matrix M with respect to eigenvalues λ and μ respectively. Prove that if $\lambda \neq \mu$ then neither $\mathbf{v}$ nor $\mathbf{w}$ can be expressed as a scalar multiple of the other, and hence $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.
- 15.* Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be eigenvectors of a square matrix M with respect to eigenvalues λ_1, λ_2 and λ_3 respectively, where λ_1, λ_2 and λ_3 are distinct. Prove that $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. (This exercise generalises to prove the theorem that any set of eigenvectors corresponding to distinct eigenvalues of a square matrix M is linearly independent.)