The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Week 9 Longer Solutions** 2020

1. Clearly every complex number is a linear combination of 1 and i, so $\{1, i\}$ spans $\mathbb{C}$. Suppose that some linear combination of 1 and i is the zero vector, in this context, say

$$0 \;=\; \lambda 1 + \mu i \;=\; \lambda + \mu i$$

where $\lambda, \mu \in \mathbb{R}$. If $\mu \neq 0$ then

$$\sqrt{-1} \;=\; i \;=\; -\frac{\lambda}{\mu} \;\in\; \mathbb{R}\,,$$

which is impossible (as square roots of negative numbers do not exist in $\mathbb{R}$). Hence $\mu = 0$, so that

$$0 \;=\; \lambda + \mu i \;=\; \lambda + 0 \;=\; \lambda\,,$$

so $\lambda = \mu = 0$. This proves $\{1, i\}$ is a linearly independent set, so forms a basis for $\mathbb{C}$.

2. Observe that the vectors in B may be identified with the rows of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

which is in row echelon form, so are linearly independent. Since B is linearly independent and has size $3 = \dim(\mathbb{R}^3)$, then B must also be a basis for $\mathbb{R}^3$. To find the coordinates of $\mathbf{v}$ in each of the three cases, we row reduce the following augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 2 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & -4 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 2 \\ 0 & 1 & 0 & -2 & -1 & -1 \\ 0 & 0 & 1 & -5 & 0 & -1 \end{array}\right]$$

The columns give the respective coordinates:

$$\text{(a)} \quad [\mathbf{v}]_B \;=\; \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} \qquad \text{(b)} \quad [\mathbf{v}]_B \;=\; \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \text{(c)} \quad [\mathbf{v}]_B \;=\; \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

3. We have

$$M \;=\; \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 1 & 3 & -4 & 6 & 5 & 4 \\ 3 & 8 & -11 & 16 & 10 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 0 & 1 & -1 & 2 & 5 & 3 \\ 0 & 2 & -2 & 4 & 10 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 0 & 1 & -1 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which has rank 2, and

$$M^\top = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 8 \\ -3 & -4 & -11 \\ 4 & 6 & 16 \\ 0 & 5 & 10 \\ 1 & 4 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 2 & 4 \\ 0 & 5 & 10 \\ 0 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which also has rank 2, as expected. The identification of a 2×3 matrix with a row vector of length 6, by concatenating the two rows of the matrix, respects addition and scalar multiplication, so if A, B and C were linearly independent then M would have rank 3. Since the rank of M is less than 3, the three matrices are linearly dependent. To find a dependency relation, we may use the reduced form of $M^\top$ above, that corresponds to a suitable homogeneous system, and we find $\lambda_1 A + \lambda_2 B + \lambda_3 C = 0$ where $\lambda_1 = -\lambda_3$ and $\lambda_2 = -2\lambda_3$. Taking, for example, $\lambda_3 = -1$, We get $A + 2B - C = 0$, so $C = A + 2B$ (and it is easy to check that this in fact holds).

4. Row reducing M, we get

$$\begin{bmatrix} 2 & 1 & 0 & -4 \\ 3 & 0 & -1 & 2 \\ 6 & -3 & -4 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 6 \\ 3 & 0 & -1 & 2 \\ 0 & -3 & -2 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 6 \\ 0 & 3 & 2 & -16 \\ 0 & -3 & -2 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 6 \\ 0 & 3 & 2 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of M is clearly 2, so any pair of linearly independent rows in any of these matrices will serve as forming a basis for the row space, for example

$$\left\{ \begin{bmatrix} 1 & -1 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 2 & -16 \end{bmatrix} \right\},$$

and any pair of linearly independent columns of M will serve as forming a basis for the column space, for example,

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \right\}.$$

Column reducing M, we get

$$\begin{bmatrix} 2 & 1 & 0 & -4 \\ 3 & 0 & -1 & 2 \\ 6 & -3 & -4 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & -1 & 2 \\ -3 & 6 & -4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -3 & 12 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 4 & 0 & 0 \end{bmatrix},$$

confirming that indeed the column rank is 2, coinciding with the row rank. To find a basis for the null space of M, we take the earlier row reduction of M further:

$$\begin{bmatrix} 1 & -1 & -1 & 6 \\ 0 & 3 & 2 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -3 & -3 & 18 \\ 0 & 3 & 2 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -1 & 2 \\ 0 & 3 & 2 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

yielding the following solution space of the associated homogeneous system, using two parameters:

$$\begin{aligned} \{(s-2t, -2s+16t, 3s, 3t) \mid s,t \in \mathbb{R}\} &= \{s(1,-2,3,0) + t(-2,16,0,3) \mid s,t \in \mathbb{R}\} \\ &= \langle (1,2,3,0), (-2,16,0,3) \rangle \,. \end{aligned}$$

Regarding this as the null space, then a basis is

$$\{(1, -2, 3, 0), (-2, 16, 0, 3)\} .$$

Regarding the null space as $\{\mathbf{v} \mid M\mathbf{v} = \mathbf{0}\}$, then a basis is formed by taking the transpose of each of these 4-tuples (regarding them as row vectors):

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix} , \begin{bmatrix} -2 \\ 16 \\ 0 \\ 3 \end{bmatrix} \right\} .$$

In either case the dimension of the null space is 2, so the nullity and rank of M are both 2, and $2 + 2 = 4$, the number of columns of M, verifying that the Rank-Nullity theorem holds in this case.

5. (a) By inspection, it is easy to see that

$$(1 + x + x^2) + (1 - x) \;=\; 2 + x^2$$

so the polynomials are linearly dependent.

(b) It is not so obvious, by inspection, whether the polynomials are linearly independent, so we row reduce the following matrix, in which the rows correspond to the respective polynomials (by taking coefficients of powers of x):

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -2 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} ,$$

which has full rank, so the polynomials are linearly independent.

(c) It is obvious that $\sin x$ and $\cos x$ are not scalar multiples of each other (from their graphs, for example), so they are linearly independent. To check this directly from the definition, suppose that

$$\lambda \sin x + \mu \cos x \;=\; 0 ,$$

the zero function, for some $\lambda, \mu \in \mathbb{R}$. Taking $x = 0$ gives $\mu = 0$. Taking $x = \pi/2$ gives $\lambda = 0$. Hence $\lambda = \mu = 0$, which verifies linear independence.

(d) Observe that $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$, so that

$$1(1) - 2(\sin^2 x) - 1(\cos 2x) \;=\; 0 ,$$

the zero function, which shows that the functions are linearly dependent.

6. Let $\alpha, \beta \in \mathbb{Q}(\sqrt{2})$. Then $\alpha = a + b\sqrt{2}$ and $\beta = c + d\sqrt{2}$ for some $a, b, c, d \in \mathbb{Q}$. Then

$$\alpha + \beta \;=\; a + c + (b + c)\sqrt{2} \;\in\; \mathbb{Q}(\sqrt{2}) ,$$

since $a + c, b + c \in \mathbb{Q}$, and

$$\alpha\beta \;=\; (a + b\sqrt{2})(c + d\sqrt{2}) \;=\; ac + 2bd + (ad + bc)\sqrt{2} \;\in\; \mathbb{Q} ,$$

since $ac+2bd, ad+bd \in \mathbb{Q}$. Suppose that $\alpha \neq 0$, so that $a \neq 0$ or $b \neq 0$. If $b = 0$ then $\alpha = a \neq 0$, so that

$$\alpha^{-1} \;=\; a^{-1} + 0\sqrt{2} \;\in\; \mathbb{Q}(\sqrt{2}) \,,$$

since $a^{-1}, 0 \in \mathbb{Q}$. Suppose then that $b \neq 0$. First observe that $a - b\sqrt{2} \neq 0$, for otherwise

$$a^2 - 2b^2 \;=\; (a+b\sqrt{2})(a-b\sqrt{2}) \;=\; (\alpha)(0) \;=\; 0 \,,$$

so that $2 = \frac{a^2}{b^2}$, whence $\sqrt{2} = \pm\frac{a}{b} \in \mathbb{Q}$, which is impossible, since $\sqrt{2} \notin \mathbb{Q}$. Hence $a - b\sqrt{2} \neq 0$ and we have

$$\alpha^{-1} \;=\; \frac{1}{a+b\sqrt{2}} \;=\; \frac{a-b\sqrt{2}}{(a+b\sqrt{2})(a-b\sqrt{2})} \;=\; \frac{a}{a^2-2b^2} + \left(\frac{-b}{a^2-2b^2}\right)\sqrt{2} \;\in\; \mathbb{Q}(\sqrt{2}) \,,$$

since $\frac{a}{a^2-2b^2}, \frac{-b}{a^2-2b^2} \in \mathbb{Q}$. This verifies all of the closure properties, so that $\mathbb{Q}(\sqrt{2})$ becomes a field, and hence also a vector space over $\mathbb{Q}$ with respect to restricted scalar multiplication. Each element of $\mathbb{Q}(\sqrt{2})$ is a linear combination of 1 and $\sqrt{2}$, so that $\{1, \sqrt{2}\}$ spans $\mathbb{Q}(\sqrt{2})$. Further $\{1, \sqrt{2}\}$ is linearly independent, for otherwise it follows that $\sqrt{2} \in \mathbb{Q}$, which is impossible. This verifies that $\{1, \sqrt{2}\}$ is a basis for $\mathbb{Q}(\sqrt{2})$ as a vector space over $\mathbb{Q}$.

7. The set

$$B \;=\; \{1, x-1, (x-1)^2\} \;=\; \{1, -1+x, 1-2x+x^2\}$$

is linearly independent, because, for example, the transpose of the matrix of coefficients

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

clearly has rank 3. Hence B, having 3 elements, must be a basis for $\mathbb{P}_2$, since 3 is the dimension of $\mathbb{P}_2$. We can find the coordinates of $p(x)$ with respect to B in each case, by row reducing the following augmented matrix, obtained by adding columns of coefficients of powers of x:

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 6 & 1 & -1 \\ 0 & 1 & -2 & -5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 2 & 0 \\ 0 & 1 & 0 & -1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array}\right]$$

The columns give the respective coordinates:

$$\text{(a)} \quad [\mathbf{v}]_B \;=\; \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \qquad \text{(b)} \quad [\mathbf{v}]_B \;=\; \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \qquad \text{(c)} \quad [\mathbf{v}]_B \;=\; \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

8. Let $B = \{\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{21}, \mathbf{e}_{22}, \mathbf{e}_{23}\}$ where

$$\mathbf{e}_{11} \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \mathbf{e}_{12} \;=\; \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \mathbf{e}_{13} \;=\; \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{e}_{21} \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; \mathbf{e}_{22} \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; \mathbf{e}_{23} \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $A \in \mathrm{Mat}_{2,3}$ then

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a_{11}\mathbf{e}_{11} + a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{21}\mathbf{e}_{21} + a_{22}\mathbf{e}_{22} + a_{23}\mathbf{e}_{23}$$

for some scalars $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$, which proves that B spans $\mathrm{Mat}_{2,3}$. Suppose that there exist scalars $\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{21}, \lambda_{22}, \lambda_{23}$ such that

$$\lambda_{11}\mathbf{e}_{11} + \lambda_{12}\mathbf{e}_{12} + \lambda_{13}\mathbf{e}_{13} + \lambda_{21}\mathbf{e}_{21} + \lambda_{22}\mathbf{e}_{22} + \lambda_{23}\mathbf{e}_{23} = \mathbf{0} ,$$

the zero vector, which is the zero matrix in this context, so that

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Then $\lambda_{11} = \lambda_{12} = \lambda_{13} = \lambda_{21} = \lambda_{22} = \lambda_{23} = 0$. This verifies that B is linearly independent, so forms a basis for $\mathrm{Mat}_{2,3}$. More generally, for $m, n \geq 1$, put

$$B = \{\mathbf{e}_{ij} \mid i = 1, \ldots, m, \;\; j = 1, \ldots, n\}$$

where each $\mathbf{e}_{ij}$ is the $m \times n$ matrix with 0's everywhere except for 1 in the (i, j)-place. If $A = [a_{ij}] \in \mathrm{Mat}_{m,n}$, with (i, j)-entry a_{ij} for each i, j then

$$A = \sum_{i,j} a_{ij}\mathbf{e}_{i,j} ,$$

which shows that B spans $\mathrm{Mat}_{m,n}$. Suppose that there exist scalars $\lambda_{i,j}$ for each i, j such that

$$\sum_{i,j} \lambda_{i,j}\mathbf{e}_{i,j} = \mathbf{0} ,$$

the zero vector, which in this context is the zero matrix. Thus the matrix whose (i, j)-entry is λ_{ij} for each i, j has zero everywhere. Thus $\lambda_{i,j} = 0$ for each i, j, which shows that B is linearly independent. Hence B is a basis for $\mathrm{Mat}_{i,j}$. Since B has mn elements, the dimension of $\mathrm{Mat}_{m,n}$ is mn.

9. (a) Working over $\mathbb{R}$ and $\mathbb{Z}_3$, using the same row reductions, we have

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

so that the rank is 3. The nullity is therefore 0, and the empty set is the basis for the null space. Working over $\mathbb{Z}_2$, however, we get

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} ,$$

so that the rank is 2 and the nullity is 1. The null space is

$$\{(t, t, t) \mid t \in \mathbb{Z}_2\} = \langle (1, 1, 1) \rangle$$

with basis $\{(1, 1, 1)\}$.

(b) Working over $\mathbb{R}$, we have

$$B = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 4 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so the rank is 3, nullity 0 and the empty set a basis for the null space. Working over $\mathbb{Z}_5$, we get

$$B = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 4 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix},$$

so the rank is 2 and the nullity 1. The null space is

$$\{(-2t, -3t, t) \mid t \in \mathbb{Z}_5\} \;=\; \{(3t, 2t, t) \mid t \in \mathbb{Z}_5\} \;=\; \langle (3, 2, 1) \rangle$$

with basis $\{(3, 2, 1)\}$.

(c) Working over $\mathbb{R}$, we have

$$C = \begin{bmatrix} -1 & 0 & 3 & -2 \\ -1 & 1 & 0 & 3 \\ -1 & 0 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & -5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

so the rank is 3 and the nullity is 1. The null space is

$$\{(t, -2t, t, t) \mid t \in \mathbb{R}\} \;=\; \langle (1, -2, 1, 1) \rangle$$

with basis $\{(1, -2, 1, 1)\}$. Working over $\mathbb{Z}_5$, we have

$$C = \begin{bmatrix} -1 & 0 & 3 & -2 \\ -1 & 1 & 0 & 3 \\ -1 & 0 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so the rank and nullity are both 2. The null space is

$$\{(3s{-}2t, 3s, s, t) \mid t \in \mathbb{Z}_5\} \;=\; \{(3s{+}3t, 3s, s, t) \mid t \in \mathbb{Z}_5\} \;=\; \langle (3, 3, 1, 0), (3, 0, 0, 1) \rangle$$

with basis $\{(3, 3, 2, 0), (3, 0, 0, 1)\}$.

10. (a) The rank of A, from the previous exercise, is 3 working over $\mathbb{R}$ and $\mathbb{Z}_3$, and 2 working over $\mathbb{Z}_2$. The size of X is 3, so X is linearly independent over $\mathbb{R}$ and $\mathbb{Z}_3$, but linearly dependent over $\mathbb{Z}_2$.

(b) The rank of B, from the previous exercise, is 3 working over $\mathbb{R}$, and 2 working over $\mathbb{Z}_5$. The size of X is 3, so X is linearly independent over $\mathbb{R}$, but linearly dependent over $\mathbb{Z}_5$.

(c) The vectors in X are obtained by transposing the columns of B, from the previous exercise. The size of X is again 3, so the conclusions are the same as the previous part, namely, X is linearly independent over $\mathbb{R}$, but linearly dependent over $\mathbb{Z}_5$.

(d) The rank of C, from the previous exercise, is 3 working over $\mathbb{R}$, and 2 working over $\mathbb{Z}_5$. The size of X is 3, so X is linearly independent over $\mathbb{R}$, but linearly dependent over $\mathbb{Z}_5$.

(e) The size of X is 4 and X is a subset of F^3, of dimension 3, so X is linearly dependent over both fields.

11. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors from a vector space V over a field F. Suppose first that they are linearly independent. If $\mathbf{v}$ is a scalar multiple of $\mathbf{w}$, say $\mathbf{v} = \lambda\mathbf{w}$ for some $\lambda \in F$, then we have

$$\mathbf{0} \;=\; \mathbf{v} - \mathbf{v} \;=\; 1\mathbf{v} - \lambda\mathbf{w} \;=\; \lambda_1\mathbf{v} + \mu\mathbf{w} \;,$$

where $\lambda_1 = 1$ and $\mu = -\lambda$, not both zero, contradicting linear independence. Hence $\mathbf{v}$ is not a scalar multiple of $\mathbf{w}$. Similarly $\mathbf{w}$ is not a scalar multiple of $\mathbf{v}$. Suppose conversely that neither vector is a scalar multiple of the other. Let $\lambda, \mu \in F$ such that

$$\lambda\mathbf{v} + \mu\mathbf{w} \;=\; \mathbf{0} \;.$$

If $\lambda \neq 0$ then, rearranging this equation we have

$$\mathbf{v} \;=\; (\frac{-\mu}{\lambda})\mathbf{w} \;,$$

so that $\mathbf{v}$ is a scalar multiple of $\mathbf{w}$, which contradicts our supposition. Hence $\lambda = 0$. A similar argument shows $\mu = 0$. Hence $\lambda = \mu = 0$, which proves $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

12. Suppose first that $\mathbf{v}_1, \ldots, \mathbf{v_n}$ are linearly independent. Suppose by way of contradiction that one of the vectors can be expressed as a linear combination of the others. Without loss of generality (reordering the vectors if necessary), we may suppose

$$\mathbf{v}_1 \;=\; \lambda_2\mathbf{v}_2 + \ldots + \lambda_n\mathbf{v}_n$$

for some scalars $\lambda_2, \ldots, \lambda_n$. But then

$$(-1)\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \ldots + \lambda_n\mathbf{v}_n \;=\; \mathbf{0} \;,$$

contradicting linear independence, since at least one nonzero scalar is used as a coefficient in this equation. This shows that no vector from our list can be expressed as a linear combination of other vectors in the list. Suppose conversely that no vector can be expressed as a linear combination of the other vectors. Let $\lambda_1, \ldots, \lambda_n$ be scalars such that

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \ldots + \lambda_n\mathbf{v}_n \;=\; \mathbf{0} \;.$$

Suppose that one of the scalars is nonzero. Without loss of generality (reordering the vectors if necessary), we may suppose that $\lambda_1 \neq 0$. Then rearranging this equation gives

$$\mathbf{v}_1 \;=\; \left(\frac{-\lambda_2}{\lambda_1}\right)\mathbf{v}_2 + \ldots + \left(\frac{-\lambda_n}{\lambda_1}\right)\mathbf{v}_n \;,$$

so that $\mathbf{v}_1$ can be expressed as a linear combination of the other vectors. This contradicts our supposition. Hence, none of the scalars are nonzero, so that $\lambda_1 = \ldots = \lambda_n = 0$, proving that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

13. Suppose that $\phi : V \to W$ is a vector space isomorphism. Let B be a basis of V. Then $\phi(B) \subseteq W$. We claim that $\phi(B)$ is a basis for W. Let $\mathbf{w} \in W$, so $\mathbf{w} = \phi(\mathbf{v})$ for some $\mathbf{v} \in V$, since ϕ is surjective. But B spans V, so

$$\mathbf{v} \;=\; \lambda_1 \mathbf{b}_1 + \ldots + \lambda_n \mathbf{b}_n$$

for some $\mathbf{b}_1, \ldots, \mathbf{b}_n \in B$ and scalars $\lambda_1, \ldots, \lambda_n$. But then, since ϕ preserves vector addition and scalar multiplication, we have

$$\mathbf{w} \;=\; \phi(\mathbf{v}) \;=\; \phi(\lambda_1 \mathbf{b}_1 + \ldots + \lambda_n \mathbf{b}_n) \;=\; \lambda_1 \phi(\mathbf{b}_1) + \ldots + \lambda_n \phi(\mathbf{b}_n) \;,$$

so that $\mathbf{w}$ is in the span of $\phi(B)$. This proves that $\phi(B)$ spans W. Let $\mathbf{w}_1, \ldots, \mathbf{w_n}$ be any distinct vectors from $\phi(B)$ and suppose that $\lambda_1, \ldots, \lambda_n$ are scalars such that

$$\lambda_1 \mathbf{w}_1 + \ldots + \lambda_n \mathbf{w}_n \;=\; \mathbf{0} \;.$$

Because ϕ is injective, there exist distinct vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in B$ such that

$$\mathbf{w}_1 \;=\; \phi(\mathbf{v}_1) \;,\; \ldots \;,\; \mathbf{w}_n \;=\; \phi(\mathbf{v}_n) \;.$$

Hence, again since ϕ preserves addition and scalar multiplication,

$$\mathbf{0} \;=\; \lambda_1 \mathbf{w}_1 + \ldots + \lambda_n \mathbf{w}_n \;=\; \lambda_1 \phi(\mathbf{v}_1) + \ldots + \lambda_n \phi(\mathbf{v}_n) \;=\; \phi(\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n) \;.$$

By injectivity again, since $\phi(\mathbf{0}) = \mathbf{0}$, we have

$$\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n \;=\; \mathbf{0} \;.$$

By linear independence of B, we conclude that $\lambda_1 = \ldots = \lambda_n = 0$. This proves $\phi(B)$ is also linearly independent, so is a basis for W. But B and $\phi(B)$ have the same size, since ϕ is bijective, so that V and W have the same dimension.

14. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for M corresponding to eigenvalues λ and μ respectively, and that $\lambda \neq \mu$. Suppose that $\mathbf{v}$ is a scalar multiple of $\mathbf{w}$, say

$$\mathbf{v} \;=\; \alpha \mathbf{w}$$

for some scalar α, which is necessarily nonzero, since $\mathbf{v}$ is a nonzero vector. Then

$$\mathbf{w} \;=\; 1\mathbf{w} \;=\; (\alpha^{-1}\alpha)\mathbf{w} \;=\; \alpha^{-1}(\alpha \mathbf{w}) \;=\; \alpha^{-1}\mathbf{v} \;.$$

Hence

$$\lambda \mathbf{v} \;=\; M\mathbf{v} \;=\; M(\alpha \mathbf{w}) \;=\; \alpha M \mathbf{w} \;=\; \alpha \mu \mathbf{w} \;=\; \alpha \mu (\alpha^{-1}\mathbf{v}) \;=\; (\alpha \alpha^{-1}) \mu \mathbf{v} \;=\; \mu \mathbf{v} \;,$$

so that

$$\mathbf{0} \;=\; \mu \mathbf{v} - \mu \mathbf{v} \;=\; \lambda \mathbf{v} - \mu \mathbf{v} \;=\; (\lambda - \mu)\mathbf{v} \;.$$

But $\mathbf{v}$ is a nonzero vector, so this forces $\lambda - \mu = 0$. But then $\lambda = \mu$, which is a contradiction. Hence $\mathbf{v}$ is not a scalar multiple of $\mathbf{w}$. If $\mathbf{w}$ is a scalar multiple of $\mathbf{v}$ then, since both vectors are nonzero, $\mathbf{v}$ is also a scalar multiple of $\mathbf{w}$, which we have just shown can't happen. Hence neither vector is a scalar multiple of the other. This proves that they are linearly independent.

15. Suppose that μ_1, μ_2, μ_3 are scalars such that

$$\mu_1 \mathbf{v}_1 + \mu_2 \mathbf{v}_2 + \mu_3 \mathbf{v}_3 \;=\; \mathbf{0} \,.$$

We will show that $\mu_1 = \mu_2 = \mu_3 = 0$. Suppose that $\mu_1 \neq 0$. Then

$$\mathbf{v}_1 \;=\; -\frac{\mu_2}{\mu_1}\mathbf{v}_2 - \frac{\mu_3}{\mu_1}\mathbf{v}_3 \,.$$

Hence, on the one hand,

$$\lambda_1 \mathbf{v}_1 \;=\; -\frac{\mu_2 \lambda_1}{\mu_1}\mathbf{v}_2 - \frac{\mu_3 \lambda_1}{\mu_1}\mathbf{v}_3 \,,$$

whilst, on the other hand, using matrix multiplication,

$$\lambda_1 \mathbf{v}_1 \;=\; M\mathbf{v}_1 \;=\; M\left(-\frac{\mu_2}{\mu_1}\mathbf{v}_2 - \frac{\mu_3}{\mu_1}\mathbf{v}_3 \right) \;=\; -\frac{\mu_2 \lambda_2}{\mu_1}\mathbf{v}_2 - \frac{\mu_3 \lambda_3}{\mu_1}\mathbf{v}_3 \,,$$

so that, after rearranging,

$$\frac{\mu_2}{\mu_1}(\lambda_1 - \lambda_2)\mathbf{v}_2 \;=\; \frac{\mu_3}{\mu_1}(\lambda_3 - \lambda_1)\mathbf{v}_3 \,.$$

But $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, so are not scalar multiples of each other, by the previous exercise, so that $\mu_2 = \mu_3 = 0$ (since $\lambda_1 - \lambda_2$ and $\lambda_3 - \lambda_1$ are nonzero, as the eigenvalues are distinct). But then $\mathbf{v}_1 = \mathbf{0}$, which contradicts that $\mathbf{v}_1$ is nonzero (being an eigenvector). This proves that $\mu_1 = 0$. Similarly $\mu_2 = \mu_3 = 0$, completing the proof that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.