Important Ideas and Useful Facts:

(i) **Matrix exponentials:** If $M$ is a real square matrix then we may form the *matrix exponential*

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \ldots .$$

It is a theorem that the series always converges. If $M$ is a diagonal $n \times n$ matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$ then $e^M$ is also diagonal with diagonal entries $e^{\lambda_1}, \ldots, e^{\lambda_n}$. If $A, B$ and $P$ are real square matrices of the same size, $P$ invertible, and $B = P^{-1}AP$ then

$$e^B = P^{-1}e^A P .$$

If $A$ and $B$ commute, that is, $AB = BA$, then $e^{A+B} = e^Ae^B$.

(ii) **Solving systems of differential equations:** Suppose that we have $n$ differentiable functions $x_1 = x_1(t), x_2 = x_2(t), \ldots, x_n = x_n(t)$ of a real variable $t$ that satisfy the following system of differential equations with constant coefficients:

$$
\begin{align*}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{align*}
$$

Put $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, so that the system may be expressed in matrix form $\mathbf{x}' = A\mathbf{x}$. The solution to this system is

$$\mathbf{x} = e^{tA}\mathbf{c}$$

where $\mathbf{c} = \mathbf{x}(0)$ is a column vector of constants.

(iii) **Linear transformations (general case):** Let $V$ and $W$ be vector spaces over a field $F$. A function $T : V \to W$ is called a *linear transformation* if $T$ respects vector addition and scalar multiplication, that is, for all $\mathbf{v}, \mathbf{w} \in V$ and $\lambda \in F$,

$$T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w}) \quad \text{and} \quad T(\lambda\mathbf{v}) = \lambda T(\mathbf{v}) ,$$

or, equivalently, $T$ preserves linear combinations, that is for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $\lambda_1, \lambda_2 \in F$,

$$T(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2) = \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2) .$$

If $V = W$ then $T$ is called a *linear operator*. If $T$ is bijective (one-one and onto) then $T$ is called a *vector space isomorphism*. The composite of linear transformations, when defined, is also a linear transformation.
(iv) Matrix of a linear transformation with respect to choice of bases: Let \( T : V \to W \) be a linear transformation, and let \( B = \{b_1, \ldots, b_n\} \) and \( D = \{d_1, \ldots, d_m\} \) be ordered bases for \( V \) and \( W \) respectively. Define the \textit{matrix of \( T \) with respect to \( B \) and \( D \)} to be

\[
[T]_B^D = \begin{bmatrix} [T(b_1)]_D & \cdots & [T(b_n)]_D \end{bmatrix},
\]

by which we mean that we write down, in order, columns of coordinates, in \( W \) with respect to \( D \), of the images under \( T \) of successive basis elements from \( B \). Note that \([T]_B^D \) is an \( m \times n \) matrix. It follows from the definitions that, for all \( v \in V \),

\[
[T(v)]_D = [T]_B^D[v]_B.
\]

(v) \textbf{The identity linear operator}: Given any vector space \( V \) the mapping \( \text{id} = \text{id}_V : V \to V \) where \( \text{id}(v) = v \), fixing all vectors in \( V \), is called the \textit{identity linear transformation} or \textit{identity operator}. If \( V \) is \( n \)-dimensional and \( B \) is any basis for \( V \) then \([\text{id}]_B^B = I_n \), the \( n \times n \) identity matrix. If \( T : V \to W \) is a linear transformations then

\[
T \circ \text{id}_V = T \quad \text{and} \quad \text{id}_W \circ T = T.
\]

Further, if \( T \) is a vector space isomorphism, so that \( T \) is invertible and \( T^{-1} : W \to V \), then

\[
T^{-1} \circ T = \text{id}_V \quad \text{and} \quad T \circ T^{-1} = \text{id}_W.
\]

(vi) \textbf{Change of basis matrix}: Let \( B \) and \( D \) be any bases for an \( n \)-dimensional vector space \( V \). The matrix \([\text{id}]_D^B \) is called a \textit{change of basis matrix} and has the effect of converting coordinates of vectors with respect to \( B \) into coordinates with respect to \( D \), in the following sense, for any vector \( v \in V \):

\[
[\text{id}]_D^B[v]_B = [v]_D.
\]

Furthermore, the change of basis matrices \([\text{id}]_D^B \) and \([\text{id}]_B^D \) are mutually inverse, that is,

\[
[\text{id}]_D^B[\text{id}]_B^D = [\text{id}]_B^D[\text{id}]_D^B = I_n.
\]

(vii) \textbf{Kernel and image of a linear transformation}: Let \( T : V \to W \) be a linear transformation. Define the \textit{kernel} of \( T \) to be \( \ker(T) = \{v \in V \mid T(v) = 0\} \), which is a subspace of \( V \), and the \textit{image} of \( T \) to be \( \text{im}(T) = \{T(v) \mid v \in V\} \), which is a subspace of \( W \).

(viii) \textbf{Criterion using the kernel for a linear transformation to be injective}: If \( T : V \to W \) is a linear transformation then \( T \) is injective (one-one) if and only if \( \ker(T) = \{0\} \).

(ix) \textbf{Rank-nullity Theorem for linear transformations}: If \( : V \to W \) is a linear transformation then \( \dim(V) = \dim(\ker(T)) + \dim(\text{im}(T)) \).
Tutorial Exercises:

1. Find the exponential matrix $e^{tA}$ where $A$ is each of the following matrices:

   (a) $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$  
   (b) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  
   (c) $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$  
   (d) $\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$

2. Solve the following systems of differential equations, where $x = x(t)$ and $y = y(t)$ are differentiable functions of a real variable $t$, with the same initial conditions $x(0) = 1$ and $y(0) = 2$ in each case:

   (a) \( x' = -x \)  
      \[ y' = 2y \]
   (b) \( x' = x + y \)  
      \[ y' = x + y \]
   (c) \( x' = x + 3y \)  
      \[ y' = 2x + 2y \]
   (d) \( x' = 5x - 6y \)  
      \[ y' = 3x - 4y \]

3. Let $B = \{(1,0), (0,1)\}$ be the standard basis for $\mathbb{R}^2$. Put

   \[ D = \{(1,1), (-1,0)\}. \]

   Explain why $D$ is a basis for $\mathbb{R}^2$ and then write down the following matrices:

   \[ A = [\text{id}]_B^D, \quad C = [\text{id}]_D^B \quad \text{and} \quad E = [\text{id}]_B^D. \]

   Now find $E^{-1}$ in the usual way and check that indeed

   \[ E^{-1} = \begin{bmatrix} (1,0)_D \\ (0,1)_D \end{bmatrix} = [\text{id}]_B^D. \]

4. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations given by the following rules:

   $f(x,y) = (x + 2y, 3x - 4y)$ and $g(x,y) = (3x - y, 2y)$.

   (a) Find each of the following, by direct calculation, where $B$ and $D$ are the bases for $\mathbb{R}^2$ in the previous exercise:

   \[ [f]_B^D, \quad [f]_D^B, \quad [g]_B^D, \quad [g]_D^B. \]

   (If you have done this correctly, you should have produced a diagonal matrix representation for $g$.)

   (b) Check, as the theory predicts, that the following equations hold:

   \[ [f]_D^B = [\text{id}]_D^B[f]_B^B[\text{id}]_B^D \quad \text{and} \quad [g]_D^B = [\text{id}]_D^B[g]_B^B[\text{id}]_B^D. \]

   (c)* Find rules for linear operators $h, k : \mathbb{R}^2 \to \mathbb{R}^2$ such that $[h]_B^B = [f]_B^D$ and $[k]_B^B = [f]_D^B$.

5. Working over $\mathbb{R}$, let $B = \{1, x, x^2\}$ be the standard basis for the vector space $\mathbb{P}_2$ of polynomials of degree at most 2. Put

   \[ D = \{1 + x^2, x + 2x^2, 1 + 2x + 3x^2\}. \]

   Explain why $D$ is a basis for $\mathbb{P}_2$ and then write down the matrix $E = [\text{id}]_B^D$. Now find $E^{-1}$ in the usual way and check that indeed

   \[ E^{-1} = \begin{bmatrix} [1]_D \\ [x]_D \\ [x^2]_D \end{bmatrix} = [\text{id}]_D^B. \]
Further Exercises:

6. Let $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ be the standard basis for $\mathbb{R}^3$. Put

$$D = \{(1, 0, 1), (1, 1, 0), (1, 1, 1)\}.$$ 

Explain why $D$ is a basis for $\mathbb{R}^3$ and then write down the matrix $E = [\text{id}]_B^D$. Now find $E^{-1}$ in the usual way and check that indeed $E^{-1} = \begin{bmatrix} (1, 0, 1) & (0, 1, 0) & (0, 0, 1) \end{bmatrix} = [\text{id}]_D^B$.

7. Find the exponential matrix $e^{tA}$ where $A$ is each of the following matrices:

   (a) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$  

   (b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$  

   (c) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

8. Solve the following systems of differential equations, where $x = x(t)$, $y = y(t)$ and $z = z(t)$ are differentiable functions of a real variable $t$, with the same initial conditions $x(0) = -1$, $y(0) = -4$ and $z(0) = 2$ in each case:

   (a) $x' = -x$  
   $y' = 2y$  
   $z' = 3z$

   (b) $x' = y - z$  
   $y' = x + z$  
   $z' = x + y$

   (c) $x' = x + y + 2z$  
   $y' = -y$  
   $z' = 2x + y + z$

9. Consider the real matrix $M = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}$.

   (a) Write down the rule for the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that the matrix of $f$ with respect to the standard bases is $M$.

   (b) Explain briefly why $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ and $D = \{(1, 3), (2, 5)\}$ are bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

   (c)* Find the matrix $[f]^B_D$ of $f$ with respect to $B$ and $D$.

10.* Let $D$ be the differential operator that takes a differentiable function to its derivative. Explain why each of the following sets is a basis of the subspace of $\mathbb{R}^R$ that it generates:

$$B_1 = \{1, x, x^2, x^3\}, \quad B_2 = \{\sin x, \cos x\}, \quad B_3 = \{e^x, e^{2x}, xe^{2x}\}.$$ 

Each of these subspaces consists of differentiable functions on which $D$ acts as an operator. Find $[D]_{B_i}^{B_j}$ for $i = 1, 2, 3$ and calculate the rank and nullity of $D$ in each case.