The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Exercises for Week 11 (beginning 11 May)** 2020

Important Ideas and Useful Facts:

(i) Inner product spaces: Let V be a vector space over $\mathbb{R}$. We call V an *inner product space* if it is equipped with an *inner product*, that is, a mapping $\langle\, ,\rangle : V \times V \to \mathbb{R}$ such that

(a) $(\forall \mathbf{u}, \mathbf{v} \in V) \quad \langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle$,

(b) $(\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V) \quad \langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$,

(c) $(\forall \mathbf{u}, \mathbf{v} \in V)(\forall \lambda \in \mathbb{R}) \quad \langle \lambda\mathbf{u}, \mathbf{v}\rangle = \lambda\langle \mathbf{u}, \mathbf{v}\rangle$,

(d) $(\forall \mathbf{v} \in V) \quad \langle \mathbf{v}, \mathbf{v}\rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v}\rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

Common examples are $V = \mathbb{R}^n$ with respect to the usual dot product, and V the vector space of continuous real functions on a closed interval $[a, b]$ with inner product defined by, for $f, g \in V$,

$$\langle f, g\rangle \;=\; \int_a^b f(x)g(x)\,dx\ .$$

(ii) Simple consequences of the inner product definition: Let V be an inner product space. Then

(e) $(\forall \mathbf{v} \in V) \quad \langle \mathbf{0}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{0}\rangle = 0$,

(f) $(\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V) \quad \langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$,

(g) $(\forall \mathbf{u}, \mathbf{v} \in V)(\forall \lambda \in \mathbb{R}) \quad \langle \mathbf{u}, \lambda\mathbf{v}\rangle = \lambda\langle \mathbf{u}, \mathbf{v}\rangle$.

(iii) Length or norm of a vector: Let V be an inner product space. Define the *length* or *norm* of a vector $\mathbf{v} \in V$ to be

$$\|\mathbf{v}\| \;=\; \langle \mathbf{v}, \mathbf{v}\rangle^{1/2}\ .$$

Length has the following properties:

(a) $(\forall \mathbf{v} \in V) \quad \|\mathbf{v}\| \geq 0$ and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$,

(b) $(\forall \lambda \in \mathbb{R})(\forall \mathbf{v} \in V) \quad \|\lambda\mathbf{v}\| = |\lambda|\,\|\mathbf{v}\|$,

(c) $(\forall \mathbf{u}, \mathbf{v} \in V) \quad \|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

The last property is known as the *triangle inequality*.

(iv) Distance between vectors: If $\mathbf{v}$ and $\mathbf{w}$ are vectors in an inner product space V then the *distance* between $\mathbf{v}$ and $\mathbf{w}$ is $\|\mathbf{v}-\mathbf{w}\| = \|\mathbf{w}-\mathbf{v}\|$. Thus, from the triangle inequality, for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

$$\|\mathbf{u}-\mathbf{w}\| \;\leq\; \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\|\ .$$

(v) The Cauchy-Schwarz inequality: If $\mathbf{u}, \mathbf{v} \in V$, where V is an inner product space, then

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \;\leq\; \|\mathbf{u}\|\|\mathbf{v}\|\ .$$

(vi) Normalising a vector: Call a vector $\mathbf{v}$ from an inner product space *normal* if $\|\mathbf{v}\| = 1$. If $\mathbf{v} \neq \mathbf{0}$ then we *normalise* $\mathbf{v}$ by forming the normal vector

$$\widehat{\mathbf{v}} \;=\; \frac{1}{\|\mathbf{v}\|}\mathbf{v}\;.$$

(vii) Orthogonal vectors: Vectors $\mathbf{u}$ and $\mathbf{v}$ from an inner product space are said to be *orthogonal* or *mutually perpendicular* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

(viii) Orthogonal and orthonormal sets of vectors: A set of vectors from an inner product space is said to be *orthogonal* if every pair of distinct vectors from the set is orthogonal. An orthogonal set in which every vector is normal is said to be *orthonormal*. It is an important fact that any orthogonal set of nonzero vectors is linearly independent.

(ix) Utility of an orthonormal basis in finding coordinates: If $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is an orthonormal basis for an inner product space V then, for all $\mathbf{v} \in V$,

$$\mathbf{v} \;=\; \langle \mathbf{v}, \mathbf{b}_1 \rangle\, \mathbf{b}_1 + \langle \mathbf{v}, \mathbf{b}_2 \rangle\, \mathbf{b}_2 + \ldots + \langle \mathbf{v}, \mathbf{b}_n \rangle\, \mathbf{b}_n\;.$$

(x) Direct sum decompositions of a vector space: Let V be a vector space. If there exists subspaces U and W of V such that

$$V \;=\; U + W \;=\; \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U\,,\; \mathbf{w} \in W\} \qquad \text{and} \qquad U \cap W \;=\; \{\mathbf{0}\}$$

then we say that V has a *direct sum decomposition with respect to U and W* and write $V = U \oplus W$. In this case, if B is a basis for U and D is a basis for W then if follows that $B \cup D$ is a basis for V.

(xi) Orthogonal complement: If W is a subspace of an inner product space V then the *orthogonal complement* of W in V is

$$W^{\perp} \;=\; \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \mid \text{for all } \mathbf{w} \in W\}\;,$$

in which case we have the direct product decomposition $V = W \oplus W^{\perp}$.

(xii) Orthogonal projection onto a subspace: Let W be a subspace of an inner product space V. The *orthogonal projection of V onto W* is the linear transformation $\mathrm{Proj} : V \to W$ that maps a vector $\mathbf{v} \in V$ to the unique vector $\mathbf{w} = \mathrm{Proj}(\mathbf{v}) \in W$ where

$$\mathbf{v} \;=\; \mathbf{w} + \mathbf{w}^{\perp}\;,$$

for some unique $\mathbf{w}^{\perp} \in W^{\perp}$ (both of which exist and are unique because $V = W \oplus W^{\perp}$). We call $\mathbf{w} = \mathrm{Proj}(\mathbf{v})$ the *projection of* $\mathbf{v}$ *on* W and $\mathbf{w}^{\perp}$ the *component of* $\mathbf{v}$ *orthogonal to* W. It is an important fact that $\mathrm{Proj}(\mathbf{v})$ is the closest vector in W to $\mathbf{v}$, that is,

$$\|\mathrm{Proj}(\mathbf{v}) - \mathbf{v}\| \;\leq\; \|\mathbf{u} - \mathbf{v}\|$$

for all $\mathbf{u} \in W$.

Tutorial Exercises:

1. Let $\mathbf{u} = (1, -3, 2)$, $\mathbf{v} = (1, 1, 0)$, $\mathbf{w} = (2, 2, -4)$. Find

(a) $-2\mathbf{w}$ (b) $\|-2\mathbf{w}\|$ (c) $\left\|\frac{-2}{\|\mathbf{w}\|}\mathbf{w}\right\|$

(d) $\mathbf{u} + \mathbf{v}$ (e) $\|\mathbf{u} + \mathbf{v}\|$ (f) $\|\mathbf{u}\| + \|\mathbf{v}\|$

Verify that the triangle inequality is holding in parts (e) and (f).

2. Let $\mathbf{u} = (2, -1, 1)$ and $\mathbf{v} = (1, 1, 2)$. Find $\mathbf{u} \cdot \mathbf{v}$ and the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

3. Let $P_0(0, 0, 0)$, $P_1(1, 1, 0)$, $P_2(1, 0, 1)$, $P_3(0, 1, 1)$ be the vertices of a tetrahedron in $\mathbb{R}^3$.

(a) Verify that the tetrahedron is regular (all faces are equilateral triangles).

(b) Find the angle θ betwen two rays joining the centre to two vertices (the "bond angle" of a methane molecule).

4. Use the dot product to verify that the angle inscribed in a semicircle is a right angle.

5. Let $\mathbf{u} = (2, 0, -1, 3)$ and $\mathbf{v} = (5, 4, 7, -1)$. Find $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\|\mathbf{u} + \mathbf{v}\|$, $\mathbf{u} \cdot \mathbf{v}$ and the angle θ between $\mathbf{u}$ and $\mathbf{v}$. Verify (as expected) that $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. Verify, however, that

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 .$$

Is this to be expected? (See Exercise **13** below.)

6. (a) Write down a vector $\mathbf{v}$, as a linear combination of $\mathbf{i}$ and $\mathbf{j}$, pointing in the direction of the line $y = 2x$ in the xy-plane.

(b) Find $\widehat{\mathbf{v}}$, the unit vector pointing in the direction of $\mathbf{v}$.

(c) Write down the position vector $\mathbf{u}$ of the point $(-1, 5)$ as a linear combination of $\mathbf{i}$ and $\mathbf{j}$.

(d) Find $\text{proj}_{\mathbf{v}}\mathbf{u} = (\mathbf{u} \cdot \widehat{\mathbf{v}})\widehat{\mathbf{v}}$, the projection of $\mathbf{u}$ in the direction of $\mathbf{v}$.

(e) Now find the distance from the point $(-1, 5)$ to the line $y = 2x$ and the nearest point on this line.

7.* Let W be the plane defined by the equation

$$x + 2y - z = 0 .$$

Let $\mathbf{b}_1 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ and $\mathbf{b}_2 = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

(a) Check that $\{\mathbf{b}_1, \mathbf{b}_2\}$ is an orthonormal basis for the subspace W.

(b) Find $\text{proj}_W\mathbf{v} = \langle \mathbf{v}, \mathbf{b}_1 \rangle \mathbf{b}_1 + \langle \mathbf{v}, \mathbf{b}_2 \rangle \mathbf{b}_2$ where $\mathbf{v} = (4, 2, -5)$.

(c) Find the distance from the point $(4, 2, -5)$ to W and the nearest point on W.

8.* Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ where $\mathbf{u} \neq \mathbf{0}$.

(a) Put $\lambda = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}$ and expand and simplify $(\lambda\mathbf{u} - \mathbf{v}) \cdot (\lambda\mathbf{u} - \mathbf{v})$.

(b) Deduce that $|\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{u}\| \, \|\mathbf{v}\|$ if and only if $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$.

Further Exercises:

9. Verify from the definition of dot product that, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$,

$$(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} \;=\; \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w} \qquad \text{and} \qquad \lambda(\mathbf{u}\cdot\mathbf{v}) \;=\; (\lambda\mathbf{u})\cdot\mathbf{v} \;=\; \mathbf{u}\cdot(\lambda\mathbf{v})\ .$$

10. Verify the identity

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 \;=\; 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

in $\mathbb{R}^n$, and interpret this geometrically in $\mathbb{R}^2$.

11. Use the Cauchy-Schwarz inequality to verify the triangle inequality in $\mathbb{R}^n$.

12. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, recall that the *distance* from $\mathbf{u}$ to $\mathbf{v}$ is

$$d(\mathbf{u},\mathbf{v}) \;=\; \|\mathbf{u}-\mathbf{v}\|\ .$$

Deduce the following triangle inequality from the usual one:

$$d(\mathbf{u},\mathbf{w}) \;\le\; d(\mathbf{u},\mathbf{v}) + d(\mathbf{v},\mathbf{w})$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

13. Let $\mathbf{v}, \mathbf{w}$ be orthogonal elements of an inner product space. Verify the so-called *Generalised Theorem of Pythagoras*:

$$\|\mathbf{v}+\mathbf{w}\|^2 \;=\; \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2\ .$$

14. Let V be an inner product space and W be the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$, that is,

$$W \;=\; \{\lambda_1\mathbf{v} +_1 +\lambda_2\mathbf{v}_2 + \ldots + \lambda_n\mathbf{v}_n \mid \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}\}\ .$$

Verify that an arbitrary vector $\mathbf{v} \in V$ is orthogonal to every vector in W if and only if $\mathbf{v}$ is orthogonal to each of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

15.* Let V be the vector space of all continuous functions: $[a,b] \to \mathbb{R}$ and define, for $f, g \in V$,

$$\langle f, g\rangle \;=\; \int_a^b f(x)g(x)\,dx\ .$$

Verify that V becomes an inner product space.

16.* In the previous exercise, take $a = -1$ and $b = 1$. Let $f, g, h \in V$ where $f(x) = 1$, $g(x) = x$ and $h(x) = x^3$ for $x \in [-1, 1]$.

(a) Find $\|f\|$, $\|g\|$, $\|h\|$, $\langle f, g\rangle$, $\langle f, h\rangle$, $\langle g, h\rangle$ and the distance between f and g. Which pairs of functions are orthogonal?

(b) More generally, let $p(x) = x^m$ and $q(x) = x^n$, where m, n are nonnegative integers. Find a simple condition on m and n characterising orthogonality of p and q in V.