1. (a) $-2w = (-4, -4, 8)$.
(b) $\| -2w \| = | -2w | = 2\|w\| = 2\sqrt{4 + 4 + 16} = 2\sqrt{24} = 4\sqrt{6}$.
(c) $\left\| \frac{-2}{\|w\|}w \right\| = \frac{2}{\|w\|}\|w\| = 2$.
(d) $u + v = (2, -2, 2)$.
(e) $\|u + v\| = \sqrt{4 + 4 + 4} = \sqrt{12} = 2\sqrt{3}$.
(f) $\|u\| + \|v\| = \sqrt{1 + 9 + 4} + \sqrt{1 + 1 + 0} = \sqrt{14} + \sqrt{2}$.

Observe that $\|u\| + \|v\| = \sqrt{14} + \sqrt{2} > 2\sqrt{2} > \sqrt{6} = 2\sqrt{3} = \|u + v\|$, so that the triangle inequality is indeed holding.

2. We have $u \cdot v = 2 - 1 + 2 = 3$ and
$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{3}{\sqrt{4 + 1 + 1 + 4}} = \frac{3}{6} = \frac{1}{2},$$
so that $\theta = \frac{\pi}{3}$.

3. (a) That the tetrahedron is regular is immediate by observing that all the lengths between vertices are the same:
$$\|\mathbf{P}_0 \mathbf{P}_1\| = \sqrt{1 + 1 + 0} = \sqrt{2}, \quad \|\mathbf{P}_0 \mathbf{P}_2\| = \sqrt{1 + 0 + 1} = \sqrt{2},$$
$$\|\mathbf{P}_0 \mathbf{P}_3\| = \sqrt{0 + 1 + 1} = \sqrt{2}, \quad \|\mathbf{P}_1 \mathbf{P}_2\| = \sqrt{0 + 1 + 1} = \sqrt{2},$$
$$\|\mathbf{P}_1 \mathbf{P}_3\| = \sqrt{1 + 0 + 1} = \sqrt{2}, \quad \|\mathbf{P}_2 \mathbf{P}_3\| = \sqrt{1 + 1 + 0} = \sqrt{2}.$$

(b) The centre of the tetrahedron is the point $Q(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and it suffices to find the angle $\theta$ between $\mathbf{QP}_0$ and $\mathbf{QF}_1$. But
$$\cos \theta = \frac{\mathbf{QF}_0 \cdot \mathbf{QF}_1}{\|\mathbf{QF}_0\|\|\mathbf{QF}_1\|} = \frac{(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \cdot (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})}{\|(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\|\|\mathbf{QF}_1\|} = \frac{-\frac{1}{4}}{\frac{3}{4}} = -\frac{1}{3},$$
so that $\theta = \cos^{-1}(-\frac{1}{3}) \approx 109.5^\circ$.

4. Let $\theta$ be the angle inscribed in a semicircle. Then $\theta$ is the angle between $u + v$ and $u - v$
for some directed line segments \( u \) and \( v \) that join the centre of the circle to its circumference, but such that \( u \neq v \) and \( u \neq -v \). But

\[
\|u\| = \|v\| = r
\]

is the radius of the circle. Then

\[
\cos \theta = \frac{(u + v) \cdot (u - v)}{\|u + v\| \|u - v\|} = \frac{u \cdot u - v \cdot v}{\|u + v\| \|u - v\|} = \frac{\|u\|^2 - \|v\|^2}{\|u + v\| \|u - v\|} = 0,
\]

so that \( \theta = \frac{\pi}{2} \).

5. We have \( \|u\| = \sqrt{4 + 0 + 1 + 9} = \sqrt{14}, \|v\| = \sqrt{25 + 16 + 49 + 1} = \sqrt{91} \),

\[
\|u + v\| = \|(7, 4, 6, 2)\| = \sqrt{49 + 16 + 36 + 4} = \sqrt{105},
\]

and \( u \cdot v = 10 + 0 - 7 - 3 = 0 \), so that \( \theta = \frac{\pi}{2} \). Observe that

\[
\|u + v\| = \sqrt{105} = \sqrt{14 + 91} < \sqrt{14 + 19 + 2\sqrt{(14)(19)}}
\]

\[
= \sqrt{(\sqrt{14} + \sqrt{91})^2} = \sqrt{14 + \sqrt{91}} = \|u\| + \|v\|,
\]

consistent with the triangle inequality. However,

\[
\|u + v\|^2 = 105 = 14 + 91 = \|u\|^2 + \|v\|^2.
\]

This is consistent with what one might expect from an analogue of the Theorem of Pythagoras, since one might expect \( u \) and \( v \) to form a “right-angled triangle”.

6. (a) We have \( v = i + 2j \).

(b) Observe that \( \|v\| = \sqrt{1 + 4} = \sqrt{5} \), so that \( \hat{v} = \frac{1}{\sqrt{5}} (i + 2j) \).

(c) The position vector of the point \((-1, 5)\) is the vector \( u = -i + 5j \).

(d) The projection of \( u \) in the direction of \( v \) is

\[
\text{proj}_v u = (u \cdot \hat{v}) \hat{v} = \frac{1}{5} (-1 + 10)(i + 2j) = \frac{9}{5} i + \frac{18}{5} j.
\]

(e) The nearest point on the line is \( \left(\frac{9}{5}, \frac{18}{5}\right) \), and the distance is

\[
\|u - \text{proj}_v u\| = \| -i + 5j - \left(\frac{9}{5} i + \frac{18}{5} j\right)\| = \| -\frac{14}{5} i + \frac{7}{5} j\|
\]

\[
= \frac{1}{5} \sqrt{196 + 49} = \frac{1}{5} \sqrt{245} = \frac{7\sqrt{5}}{5}.
\]

7. (a) Clearly neither \( b_1 \) nor \( b_2 \) is a scalar multiple of the other and both satisfy the equation defining \( W \). Hence they form a basis for \( W \) (since \( W \) is two-dimensional). It suffices then to check that \( b_1 \) and \( b_2 \) are orthogonal and have length 1. The first follows because

\[
b_1 \cdot b_2 = -\frac{1}{\sqrt{6}} + 0 + \frac{1}{\sqrt{6}} = 0,
\]

and the second because

\[
b_1 \cdot b_1 = \frac{1}{2} + 0 + \frac{1}{2} = 1 \quad \text{and} \quad b_2 \cdot b_2 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.
\]
(b) We have
\[
\text{proj}_W \mathbf{v} = (\mathbf{v} \cdot \mathbf{b}_1)\mathbf{b}_1 + (\mathbf{v} \cdot \mathbf{b}_2)\mathbf{b}_2 = -\frac{1}{\sqrt{2}}\mathbf{b}_1 - \frac{7}{\sqrt{3}}\mathbf{b}_2
\]
\[
= (-\frac{1}{2}, 0, -\frac{1}{2}) + (\frac{7}{3}, -\frac{7}{3}, -\frac{7}{3}) = (\frac{11}{6}, -\frac{7}{3}, -\frac{17}{6}) .
\]
(c) Hence the closest point on \(W\) to \((4, 2, -5)\) is \((\frac{11}{6}, -\frac{7}{3}, -\frac{17}{6})\) and the distance is
\[
\|\mathbf{u} - \text{proj}_W \mathbf{u}\| = \|(4, 2, -5) - (\frac{11}{6}, -\frac{7}{3}, -\frac{17}{6})\| = \|(\frac{7}{6}, \frac{5}{3}, \frac{5}{6})\| = \frac{13\sqrt{6}}{6}.
\]
8. (a) Observe that
\[
(\lambda \mathbf{u} - \mathbf{v}) \cdot (\lambda \mathbf{u} - \mathbf{v}) = \lambda^2 \mathbf{u} \cdot \mathbf{u} - 2\lambda \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}
\]
\[
= \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2} + \mathbf{v} \cdot \mathbf{v}
\]
\[
= \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2} + \mathbf{v} \cdot \mathbf{v}
\]
\[
= \|\mathbf{v}\|^2 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2}.
\]
(b) If \(\mathbf{v}\) is a scalar multiple of \(\mathbf{u}\), say \(\mathbf{v} = \mu \mathbf{u}\) for some scalar \(\mu\), then
\[
|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u} \cdot \mu \mathbf{u}| = |\mu| |\mathbf{u} \cdot \mathbf{u}| = |\mu| \|\mathbf{u}\|^2 = \|\mathbf{u}\| |\mu| \|\mathbf{u}\| = \|\mathbf{u}\| \|\mathbf{v}\|.
\]
Conversely, suppose that \(\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\|\). Since \(\mathbf{u} \neq \mathbf{0}\), we may consider the scalar \(\lambda = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\). By part (a),
\[
\|\lambda \mathbf{u} - \mathbf{v}\|^2 = (\lambda \mathbf{u} - \mathbf{v}) \cdot (\lambda \mathbf{u} - \mathbf{v}) = \|\mathbf{v}\|^2 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2}
\]
\[
= \|\mathbf{v}\|^2 - \frac{(\|\mathbf{u}\|^2 \|\mathbf{v}\|^2)}{\|\mathbf{u}\|^2} = \|\mathbf{v}\|^2 - \|\mathbf{v}\|^2 = 0,
\]
so that \(\lambda \mathbf{u} - \mathbf{v} = \mathbf{0}\). Hence \(\mathbf{v} = \lambda \mathbf{u}\), so that \(\mathbf{v}\) is a scalar multiple of \(\mathbf{u}\).

9. Let \(\mathbf{u} = (u_1, \ldots, u_n)\), \(\mathbf{v} = (v_1, \ldots, v_n)\), \(\mathbf{w} = (w_1, \ldots, w_n)\) \(\in \mathbb{R}^n\) and \(\lambda \in \mathbb{R}\). Then
\[
(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (u_1 + v_1, \ldots, u_n + v_n) \cdot (w_1, \ldots, w_n)
\]
\[
= (u_1 + v_1)w_1 + \ldots + (u_n + v_n)w_n
\]
\[
= u_1w_1 + v_1w_1 + \ldots + u_nw_n + v_nw_n
\]
\[
= u_1w_1 + \ldots + u_nw_n + v_1w_1 + \ldots + v_nw_n
\]
\[
= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w},
\]
and
\[
\lambda(\mathbf{u} \cdot \mathbf{v}) = \lambda(u_1v_1 + \ldots + u_nv_n)
\]
\[
= (\lambda u_1)v_1 + \ldots + (\lambda u_n)v_n = u_1(\lambda v_1) + \ldots + u_n(\lambda v_n)
\]
\[
= (\lambda u_1, \ldots, \lambda u_n) \cdot (v_1, \ldots, v_n) = (u_1, \ldots, u_n) \cdot (\lambda v_1, \ldots, \lambda v_n)
\]
\[
= (\lambda \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\lambda \mathbf{v}).
\]
10. We have

\[\|u + v\|^2 + \|u - v\|^2 = (u + v) \cdot (u + v) + (u - v) \cdot (u - v)\]
\[= u \cdot u + 2u \cdot v + v \cdot v + u \cdot u - 2u \cdot v + v \cdot v\]
\[= 2u \cdot u + 2v \cdot v = 2\|u\|^2 + 2\|v\|^2.\]

Interpreted in \(\mathbb{R}^2\), this result says that the sum of the squares of the diagonals of a parallelogram add up to the sum of the squares of the lengths of the sides.

11. For \(u, v \in \mathbb{R}^n\), the Cauchy-Schwarz inequality gives

\[\|u + v\|^2 = (u + v) \cdot (u + v) = u \cdot u + 2u \cdot v + v \cdot v\]
\[\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2\]

so that, taking square roots of nonnegative numbers,

\[\|u + v\| \leq \|u\| + \|v\|,\]

which is the triangle inequality.

12. Using the triangle inequality, we have, for all \(u, v, w \in \mathbb{R}^n\),

\[d(u, w) = \|u - w\| = \|u - v + v - w\| \leq \|u - v\| + \|v - w\| = d(u, v) + d(v, w).\]

13. Since \(v\) and \(w\) are orthogonal, we have \(\langle v, w \rangle = 0\). Hence, by properties of the inner product,

\[\|v + w\|^2 = \langle v + w, v + w \rangle = \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle\]
\[= \langle v, v \rangle + 0 + 0 + \langle w, w \rangle = \|v\|^2 + \|w\|^2.\]

14. If \(v\) is orthogonal to everything in \(W\) then, in particular, \(v\) is orthogonal to each of \(v_1, \ldots, v_n\), which are clearly elements of \(W\). Suppose, conversely, that \(v\) is orthogonal to each of \(v_1, \ldots, v_n\). Let \(w\) be any element of \(W\), so

\[w = \lambda_1 v_1 + \ldots + \lambda_n v_n\]

for some \(\lambda_1, \ldots, \lambda_n \in \mathbb{R}\). Then, by properties of the inner product,

\[\langle v, w \rangle = \lambda_1 \langle v, v_1 \rangle + \ldots + \lambda_n \langle v, v_n \rangle = \lambda_1 0 + \ldots + \lambda_n 0 = 0,\]

so that \(v\) is orthogonal to \(w\).

15. Let \(f, g, h \in V\) and \(\lambda \in \mathbb{R}\). Then

\[\langle f, g \rangle = \int_a^b f(x)g(x) \, dx = \int_a^b g(x)f(x) \, dx = \langle g, f \rangle,\]

\[\langle f + g, h \rangle = \int_a^b (f(x) + g(x))h(x) \, dx = \int_a^b f(x)h(x) + g(x)h(x) \, dx\]
\[= \int_a^b f(x)h(x) \, dx + \int_a^b g(x)h(x) \, dx = \langle f, h \rangle + \langle g, h \rangle,\]

4
\[
\langle \lambda f, g \rangle = \int_a^b \lambda f(x)g(x) \, dx = \lambda \int_a^b f(x)g(x) \, dx = \lambda \langle f, g \rangle,
\]
and
\[
\langle f, f \rangle = \int_a^b f(x)f(x) \, dx = \int_a^b (f(x))^2 \, dx \geq 0,
\]
since, in this last case, the integrand is nonnegative throughout the interval. Certainly
\[
\langle 0, 0 \rangle = \int_a^b 0(x)0(x) \, dx = \int_a^b 0 \, dx = 0.
\]
Suppose that \( f \in V \) such that \( \langle f, f \rangle = 0 \), so that \( \int_a^b (f(x))^2 \, dx = 0 \). We show that \( f = 0 \). Suppose to the contrary that \( f \neq 0 \). Then \( f(x) \neq 0 \) for some \( x \in [a, b] \). By continuity of \( f \), there is some \( \varepsilon > 0 \) and some interval \( I \) of positive width \( \delta > 0 \) such that
\[
|f(y)| \geq \varepsilon \quad (\forall y \in I).
\]
But then
\[
\int_a^b (f(x))^2 \, dx \geq \varepsilon^2 \delta > 0,
\]
which contradicts that \( \int_a^b (f(x))^2 \, dx = 0 \). Hence \( f = 0 \). This proves that \( \langle f, f \rangle = 0 \) if and only if \( f = 0 \), and completes the verification that \( V \) is an inner product space.

16. (a) We have
\[
\langle f, f \rangle = \int_{-1}^1 x^2 \, dx = 2, \quad \langle g, g \rangle = \int_{-1}^1 x^2 \, dx = \frac{2}{3}, \quad \langle h, h \rangle = \int_{-1}^1 x^6 \, dx = \frac{2}{7},
\]
so that \( \|f\| = \sqrt{2}, \|g\| = \sqrt{\frac{2}{3}} \) and \( \|h\| = \sqrt{\frac{2}{7}} \). Further
\[
\langle f, g \rangle = \int_{-1}^1 x \, dx = 0, \quad \langle f, h \rangle = \int_{-1}^1 x^3 \, dx = 0, \quad \langle g, h \rangle = \int_{-1}^1 x^4 \, dx = \frac{2}{5}.
\]
Observe also that
\[
\|f - g\|^2 = \int_{-1}^1 (1 - x)^2 \, dx = \left[ -\frac{(1 - x)^3}{3} \right]_{-1}^1 = \frac{8}{3},
\]
so that the distance between \( f \) and \( g \) is \( \sqrt{\frac{8}{3}} = 2\sqrt{\frac{2}{3}} \). The pairs \( f, g \) and \( f, h \) are orthogonal.

(b) We have \( \langle p, q \rangle = 0 \) if and only if \( \int_{-1}^1 x^{m+n} \, dx = 0 \), which occurs if and only if
\[
\left[ \frac{x^{m+n+1}}{m+n+1} \right]_{-1}^1 = 0,
\]
which, in turn, occurs if and only if \( 1^{m+n+1} = (-1)^{m+n+1} \), that is, \( m + n + 1 \) is even, or equivalently, \( m + n \) is odd. Thus \( p \) and \( q \) are orthogonal if and only if \( m \) and \( n \) have different parity.