1. (a) We have

\[ W = \{ (x, y, z) \mid x + 2y - z = 0 \} = \{ (-2y + z, y, z) \mid y, z \in \mathbb{R} \} = \{ y(-2, 1, 0) + z(1, 0, 1) \mid y, z \in \mathbb{R} \} = \{ \lambda_1(1, 0, 1) + \lambda_2(-2, 1, 0) \mid \lambda_1, \lambda_2 \in \mathbb{R} \} = \{ \lambda_1v_1 + \lambda_2v_2 \mid \lambda_1, \lambda_2 \in \mathbb{R} \} = \langle v_1, v_2 \rangle, \]

which verifies that \( v_1 \) and \( v_2 \) span \( W \).

(b) First stage: observe that \( \|v_1\| = \sqrt{1 + 0 + 0} = \sqrt{2} \), so that

\[ b_1 = \hat{v}_1 = \frac{1}{\sqrt{2}}(1, 0, 1). \]

Second stage:

\[ w_2 = v_2 - (v_2 \cdot b_1)b_1 = (-2, 1, 0) - \frac{-2}{2}(1, 0, 1) = (-1, 1, 1), \]

\[ b_2 = \hat{w}_2 = \frac{1}{\sqrt{3}}(-1, 1, 1). \]

Thus an orthonormal basis for \( W \) is \( \left\{ \frac{1}{\sqrt{2}}(1, 0, 1), \frac{1}{\sqrt{3}}(-1, 1, 1) \right\} \).

2. (a) We have

\[ W = \{ (x, y, z, w) \mid x - y + z - w = 0 \} = \{ (y - z + w, y, z, w) \mid y, z, w \in \mathbb{R} \} = \{ y(1, 1, 0, 0) + z(-1, 0, 1, 0) + w(1, 0, 0, 1) \mid y, z, w \in \mathbb{R} \} = \{ \lambda_1v_1 + \lambda_2v_2 + \lambda_3v_3 \mid \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \} = \langle v_1, v_2, v_3 \rangle, \]

which verifies that \( v_1, v_2 \) and \( v_3 \) span \( W \).

(b) First stage: observe that \( \|v_1\| = \sqrt{1 + 0 + 0 + 0} = \sqrt{2} \), so that

\[ b_1 = \hat{v}_1 = \frac{1}{\sqrt{2}}(1, 1, 0, 0). \]

Second stage:

\[ w_2 = v_2 - (v_2 \cdot b_1)b_1 = (-1, 0, 1, 0) - \frac{-1}{2}(1, 1, 0, 0) = \frac{1}{2}(-1, 1, 2, 0), \]

\[ b_2 = \hat{w}_2 = \frac{1}{\sqrt{6}}(-1, 1, 2, 0). \]
3. Let \( v_n \) properties of the inner product and by orthogonality, such that \( \lambda = \frac{1}{2} (1, 1, 0, 0) - \frac{1}{6} (-1, 1, 2, 0) = \frac{1}{3} (1, -1, 1, 3) \),
\[
b_3 = \frac{1}{2\sqrt{3}} (1, -1, 1, 3).
\]
Thus an orthonormal basis for \( W \) is
\[
\left\{ \frac{1}{\sqrt{2}} (1, 1, 0, 0), \frac{1}{\sqrt{6}} (-1, 1, 2, 0), \frac{1}{2\sqrt{3}} (1, -1, 1, 3) \right\}.
\]
(c) We have
\[
\text{proj}_W v = (v \cdot b_1) b_1 + (v \cdot b_2) b_2 + (v \cdot b_3) b_3
\]
\[
= \frac{3}{2} (1, 1, 0, 0) + \frac{7}{6} (-1, 1, 2, 0) + \frac{14}{12} (1, -1, 1, 3)
\]
\[
= \left( \frac{3}{2}, \frac{3}{2}, \frac{7}{2}, \frac{7}{2} \right).
\]
(d) Thus the closest point on \( W \) to \( v \) is \( \left( \frac{3}{2}, \frac{3}{2}, \frac{7}{2}, \frac{7}{2} \right) \), and the shortest distance is
\[
\| (1, 2, 3, 4) - \left( \frac{3}{2}, \frac{3}{2}, \frac{7}{2}, \frac{7}{2} \right) \| = \| (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \| = 1.
\]
3. The Generalised Theorem of Pythagoras states that if \( v_1, \ldots, v_n \) are pairwise orthogonal vectors in an inner product space, where \( n \geq 2 \), then
\[
\| v_1 + \ldots + v_n \|^2 = \| v_1 \|^2 + \ldots + \| v_n \|^2.
\]
(a) We have
\[
\| 3b_1 + 4b_2 \| = \sqrt{\| 3b_1 \|^2 + \| 4b_2 \|^2} = \sqrt{9 \| b_1 \|^2 + 16 \| b_2 \|^2} = \sqrt{25} = 5.
\]
(b) We have
\[
\| b_1 + 2b_2 - 2b_3 \| = \sqrt{\| b_1 \|^2 + \| 2b_2 \|^2 + \| -2b_3 \|^2}
\]
\[
= \sqrt{1 + 4 \| b_2 \|^2 + 4 \| b_3 \|^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3.
\]
(c) We have
\[
\| 4b_1 - b_2 + 8b_3 \| = \sqrt{\| 4b_1 \|^2 + \| -b_2 \|^2 + \| 8b_3 \|^2} = \sqrt{16 \| b_1 \|^2 + \| b_2 \|^2 + 64 \| b_3 \|^2}
\]
\[
= \sqrt{16 + 1 + 64} = \sqrt{81} = 9.
\]
4. Let \( v_1, \ldots, v_n \in X \) be pairwise orthogonal nonzero vectors. Suppose \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) such that \( \lambda_1 v_1 + \ldots + \lambda_n v_n = 0 \). Then, for each \( i = 1, \ldots, n \), we have, by elementary properties of the inner product and by orthogonality,
\[
0 = \langle 0, v_i \rangle = \langle \lambda_1 v_1 + \ldots + \lambda_n v_n, v_i \rangle
\]
\[
= \lambda_1 \langle v_1, v_i \rangle + \ldots + \lambda_{i-1} \langle v_{i-1}, v_i \rangle + \lambda_i \langle v_i, v_i \rangle + \lambda_{i+1} \langle v_{i+1}, v_i \rangle + \ldots + \lambda_n \langle v_n, v_i \rangle
\]
\[
= \lambda_1 (0) + \ldots + \lambda_{i-1} (0) + \lambda_i (v_i) + \lambda_{i+1} (0) + \ldots + \lambda_n (0)
\]
\[
= \lambda_i \| v_i \|^2,
\]
since \( \lambda_i = 0 \), since \( \lambda_i \| v_i \| \neq 0 \), as \( v_i \) is nonzero. This shows that \( \lambda_1 = \ldots = \lambda_n = 0 \), and completes the proof that \( X \) is linearly independent.
5. (a) We have

\[
\| \cos x - \sin x \|^2 = \int_{-\pi}^{\pi} (\cos x - \sin x)^2 \, dx = \int_{-\pi}^{\pi} \cos^2 x - 2 \cos x \sin x + \sin^2 x \, dx
\]

\[
= \int_{-\pi}^{\pi} 1 - 2 \sin 2x \, dx = \left[ x + \frac{\cos 2x}{2} \right]_{-\pi}^{\pi} = 2\pi ,
\]

so that the distance from \( \cos x \) to \( \sin x \) in \( V \) is \( \sqrt{2\pi} \).

(b) We have \( \| \cos 0x \|^2 = \int_{-\pi}^{\pi} 1 \, dx = 2\pi \), so \( \| \cos 0x \| = \sqrt{2\pi} \). For \( n \geq 1 \),

\[
\| \cos nx \|^2 = \int_{-\pi}^{\pi} \cos^2 nx \, dx = \int_{-\pi}^{\pi} \frac{1 + \cos 2nx}{2} \, dx = \left[ \frac{x}{2} + \frac{\sin 2nx}{4n} \right]_{-\pi}^{\pi} = \pi
\]

and

\[
\| \sin nx \|^2 = \int_{-\pi}^{\pi} \sin^2 nx \, dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2nx}{2} \, dx = \left[ \frac{x}{2} - \frac{\sin 2nx}{4n} \right]_{-\pi}^{\pi} = \pi ,
\]

so that \( \| \cos nx \| = \| \sin nx \| = \sqrt{\pi} \).

(c) Let \( m, n \geq 1, m \neq n \). Then

\[
\langle \cos 0x, \cos nx \rangle = \int_{-\pi}^{\pi} \cos 0x \cos nx \, dx = \left[ \sin nx \right]_{-\pi}^{\pi} = 0 ,
\]

and

\[
\langle \cos mx, \cos nx \rangle = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(m + n)x + \cos(m - n)x \, dx
\]

\[
= \frac{1}{2} \left[ \sin(m + n)x \, m + n + \sin(m - n)x \, m - n \right]_{-\pi}^{\pi} = 0 .
\]

(d) Let \( m, n \geq 1, m \neq n \). Then

\[
\langle \sin mx, \sin nx \rangle = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(m - n)x - \cos(m + n)x \, dx
\]

\[
= \frac{1}{2} \left[ \sin(m - n)x \, m - n - \sin(m + n)x \, m + n \right]_{-\pi}^{\pi} = 0 .
\]

(e) If \( n \geq 0 \) and \( m \geq 1 \) then

\[
\langle \cos nx, \sin mx \rangle = \int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0 ,
\]

since the integrand is an odd function.

(f) By the Generalised Theorem of Pythagoras, for \( n \geq 1 \), since the functions are orthogonal,

\[
\| \cos nx - \sin nx \|^2 = \| \cos nx \|^2 + \| \sin nx \|^2 = \pi + \pi = 2\pi ,
\]

so the distance is \( \sqrt{2\pi} \).
6. (a) First stage:

\[ b_1 = \hat{v}_1 = \frac{1}{\sqrt{2}}(-1, 1, 0, 0). \]

Second stage:

\[ w_2 = v_2 - (v_2 \cdot b_1)b_1 = (-2, 0, 1, 0) - \frac{2}{\sqrt{2}}(-1, 1, 0, 0) = (-1, -1, 1, 0), \]

\[ b_2 = \hat{w}_2 = \frac{1}{\sqrt{3}}(-1, -1, 1, 0). \]

Third stage:

\[ w_3 = v_3 - (v_3 \cdot b_1)b_1 - (v_3 \cdot b_2)b_2 \]

\[ = (1, 0, 0, 1) - \frac{1}{2}(-1, 1, 0, 0) - \frac{1}{3}(-1, -1, 1, 0) = \frac{1}{6}(1, 1, 2, 6), \]

\[ b_3 = \hat{w}_3 = \frac{1}{\sqrt{42}}(1, 1, 2, 6). \]

Thus an orthonormal basis for \( W \) is

\[ \left\{ \frac{1}{\sqrt{2}}(-1, 1, 0, 0), \frac{1}{\sqrt{3}}(-1, -1, 1, 0), \frac{1}{\sqrt{42}}(1, 1, 2, 6) \right\}. \]

(b) We have

\[ \text{proj}_W v = (v \cdot b_1)b_1 + (v \cdot b_2)b_2 + (v \cdot b_3)b_3 \]

\[ = \frac{2}{2}(-1, 1, 0, 0) + \frac{1}{3}(-1, -1, 1, 0) + \frac{14}{42}(1, 1, 2, 6) \]

\[ = (-1, 1, -1, -2). \]

(c) Thus the closest point on \( W \) to \( v \) is \((-1, 1, -1, -2)\), and the shortest distance is

\[ \| (0, 2, 1, -3) - (-1, 1, -1, -2) \| = \| (1, 1, 2, -1) \| = \sqrt{7}. \]

7. Put \( v_1 = 1, v_2 = x, v_3 = x^2 \) and \( v_4 = x^3 \). Notice that \( \{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\} \) and \( \{v_3, v_4\} \) are orthogonal pairs, since the sum of the exponents as powers of \( x \) is odd in each case. This simplifies the following calculations.

First stage: observe that \( \| 1 \| = \sqrt{\int_{-1}^{1} 1^2 \, dx} = \sqrt{2} \), so that

\[ b_1 = \hat{v}_1 = \frac{1}{\sqrt{2}}. \]

Second stage:

\[ w_2 = v_2 - (v_2 \cdot b_1)b_1 = v_2, \]

\[ b_2 = \hat{w}_2 = \frac{x}{\| x \|} = \frac{x}{\sqrt{\int_{-1}^{1} x^2 \, dx}} = \frac{x}{\sqrt{2/3}} = \sqrt{\frac{3}{2}} x. \]
Thus an orthonormal basis for $W$ is

$$
\left\{ \frac{1}{\sqrt{2}}, \frac{\sqrt{3} x}{\sqrt{2}}, \frac{\sqrt{5}}{2\sqrt{2}} (3x^2 - 1), \frac{\sqrt{7}}{2\sqrt{2}} (5x^3 - 3x) \right\}.
$$

8. (a) Observe first that $y = f(x) \cos mx$ is an odd function for $x \in (-\pi, \pi)$ and $m \geq 0$, so that

$$
a_0 = a_1 = \ldots = a_k = 0.
$$

Observe next that $y = f(x) \sin nx$ is an even function for $x \in (-\pi, \pi)$ and $n \geq 1$, so that

$$
b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx
$$

$$
= \frac{2}{\pi} \left[ - \frac{x \cos nx}{n} \right]_0^\pi - \frac{2}{\pi} \int_0^\pi \frac{\cos nx}{n} \, dx
$$

$$
= - \frac{2 \cos n\pi}{n} + \frac{2}{n^2\pi} \left[ \sin nx \right]_0^\pi
$$

$$
= \frac{2}{n}(-1)^{n-1}.
$$

Thus

$$
\text{proj}_{W_k} f = 2 \sum_{n=1}^k \frac{(-1)^{n-1}}{n} \sin nx
$$

$$
= 2 \left( \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots + \frac{(-1)^{k-1} \sin kx}{k} \right).
$$
(b) Letting $k \to \infty$ gives the Fourier series

$$f(x) = 2 \left( \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots \right).$$

Putting $x = \pi/2$ gives

$$\frac{\pi}{2} = 2 \left( 1 - \frac{0}{2} + \frac{-1}{3} - \frac{0}{4} + \frac{1}{5} - \frac{0}{6} + \frac{-1}{7} + \ldots \right),$$

so that, finally,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$  

9. (a) Let $h$ be defined by $h(x) = f(x)g(x)$. Then $h$ is odd, being a product of an odd and an even function, so that

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx = \int_{-\pi}^{\pi} h(x) \, dx = 0,$$

so that $f$ and $g$ are orthogonal.

(b) Observe that $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \, dx = \pi^2/3$, and, for $n \geq 1$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{1}{\pi} \left[ \frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_{-\pi}^{\pi} = \frac{4}{n^2} (-1)^n,$$

and $b_n = 0$, since $f$ is even and the sine function is odd. Hence

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx.$$  

(c) From part (b), in particular,

$$\pi^2 = f(\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos n\pi = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2},$$

giving $\frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$, so that, finally,

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots.$$  

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