MATH2022 Week 02

Worksheet Solutions

MATH 2022 Week 2 Worksheet

Q1/ Complete the following tables:

$\mathbb{Z}_4$:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\mathbb{Z}_5$:

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$\mathbb{Z}_6$:

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

$\mathbb{Z}_7$:

$\cdot$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Q2/ Give a brief reason why each of the following arithmetics is <u>not</u> a field.

	Reason
$\mathbb{N}$	2 does not have an additive inverse
$\mathbb{Z}$	2 does not have a multiplicative inverse
$\mathbb{Z}_4$	$2 \times 2 = 0$ but $2 \neq 0$
$\mathbb{Z}_6$	$2 \times 3 = 0$ but $2, 3 \neq 0$
$\mathbb{Z}_{10}$	$2 \times 5 = 0$ but $2, 5 \neq 0$
$\mathrm{Mat}_2(\mathbb{R})$	multiplication is not commutative e.g. $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}\neq\begin{bmatrix}0&0\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}$

Q3/

(a) In $\mathbb{Z}_5$, $-1 = 4$ and $-2 = \boxed{3}$,

$2 \times 3 = 1$, so $\frac{1}{2} = \boxed{3}$ and $\frac{1}{3} = \boxed{2}$.

(b) In $\mathbb{Z}_7$, $-1 = \boxed{6}$ and $-2 = \boxed{5}$,

$3 \times 4 = 5$, so $\frac{5}{3} = \boxed{4}$ and $\frac{5}{4} = \boxed{3}$.

(c) Evaluate in $\mathbb{Z}_{13}$:

$$\frac{(2 \times 6) - (3 \times 7)}{4^4} = \boxed{\frac{12 - 21}{16 \times 16} = \frac{-9}{3 \times 3} = \frac{-9}{9} = -1 = 12}$$

(d) In $\mathbb{Z}_{24}$,

$$16 \times 16 = (-8) \times (-8) = \boxed{64 = 16}$$

so $16^k = \boxed{16}$ for all $k \in \mathbb{Z}^+$.

Q4/ The associative law for $+$ says

$$(\forall a, b, c) \quad a+(b+c) = \boxed{(a+b)+c}.$$

The commutative law for $\cdot$ says

$$(\forall a, b) \quad a \cdot b = \boxed{b \cdot a}.$$

The distributive law says

$$(\forall a, b, c) \quad (a+b)c = \boxed{ac + bc}.$$

Let G be a group with identity e.
Then G is abelian if

$$\boxed{(\forall g, h \in G) \quad gh = hg}$$

The identity element e has the property

$$(\forall g \in G) \quad ge = eg = \boxed{g}.$$

If $g \in G$ then $gg^{-1} = g^{-1}g = \boxed{e}$.

Q5/ Let G be a group and $g, h \in G$.

Then $$(gh)^{-1} = h^{-1}g^{-1}$$

Proof :

$$(gh)(h^{-1}g^{-1}) = ((gh)h^{-1})g^{-1}$$
$$= (g(hh^{-1}))g^{-1}$$

by $\boxed{\text{associativity}}$

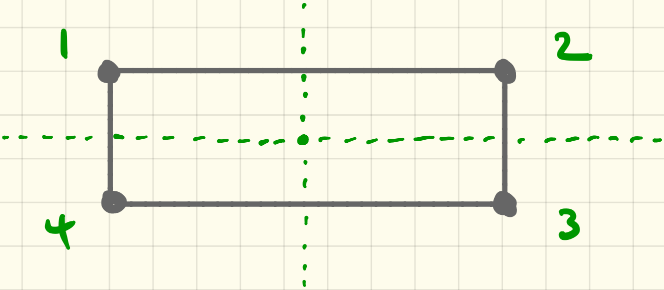
$$= (ge)g^{-1} = gg^{-1} = e,$$

and

$$(h^{-1}g^{-1})(gh) =$$

$$\boxed{\begin{aligned} h^{-1}(g^{-1}(gh)) &= h^{-1}((g^{-1}g)h) \\ &\quad \text{by associativity} \\ &= h^{-1}(eh) = h^{-1}h = e. \end{aligned}}$$

Q6/



Put α = rotation 180° about centre,
so

$\alpha : 1 \mapsto \boxed{3},\ 2 \mapsto \boxed{4},\ 3 \mapsto \boxed{1},\ 4 \mapsto \boxed{2}.$

Put β = reflection in vertical axis of symmetry,
so

$\beta : 1 \mapsto \boxed{2},\ 2 \mapsto \boxed{1},\ 3 \mapsto \boxed{4},\ 4 \mapsto \boxed{3}.$

Put γ = reflection in horizontal axis of symmetry,
so

$\gamma : 1 \mapsto \boxed{4},\ 2 \mapsto \boxed{3},\ 3 \mapsto \boxed{2},\ 4 \mapsto \boxed{1}.$

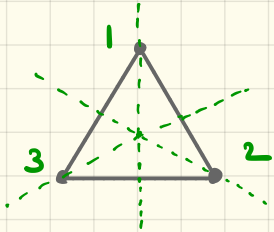
Complete the table:

	1	α	β	γ
1	1	α	β	γ
α	α	1	γ	β
β	β	γ	1	α
γ	γ	β	α	1

The group is abelian?

True False

Q7/



Put

α = rotation 120° clockwise

so

$\alpha : 1 \mapsto \boxed{2}, \; 2 \mapsto \boxed{3}, \; 3 \mapsto \boxed{1}.$

Put β = reflection in vertical axis of symmetry,

so

$\beta : 1 \mapsto \boxed{1}, \; 2 \mapsto \boxed{3}, \; 3 \mapsto \boxed{2}.$

Then

$\alpha^2 : 1 \mapsto 3, \; 2 \mapsto 1, \; 3 \mapsto 2$

$\alpha\beta : 1 \mapsto 3, \; 2 \mapsto 2, \; 3 \mapsto 1$

$\alpha^2\beta : 1 \mapsto 2, \; 2 \mapsto 1, \; 3 \mapsto 3$

Complete the table:

	1	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
1	1	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	1	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	1	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	1	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	1	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	1

The group is abelian?

True False

Q8/ Working over $\mathbb{Z}_2$, put

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A^2B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Complete the table:

	I	A	A^2	B	AB	A^2B
I	I	A	A^2	B	AB	A^2B
A	A	A^2	I	AB	A^2B	B
A^2	A^2	I	A	A^2B	B	AB
B	B	A^2B	AB	I	A^2	A
AB	AB	B	A^2B	A	I	A^2
A^2B	A^2B	AB	B	A^2	A	I

How does this compare with Q7?

Same table if $A \equiv \alpha$, $B \equiv \beta$, $I \equiv 1$