MATH2022 Week 03 Worksheet Solutions

MATH 2022 Week 3 Worksheet

Q1/ The determinant of a 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$\det M = ad - bc.$$

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} -3 & 4 \\ -5 & 7 \end{bmatrix}$$

Working over $\mathbb{R}$, find

$$\det A = \boxed{4-6 = -2}$$

$$\det B = \boxed{6-6 = 0}$$

$$\det C = \boxed{-21-(-20) = -1}$$

Which matrix is not invertible? $\boxed{B}$

Find

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$C^{-1} = \frac{1}{-1}\begin{bmatrix} 7 & -4 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ -5 & 3 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} -13 & 18 \\ -29 & 40 \end{bmatrix}$$

$$A^{-1}C^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} -7 & 4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -5 \\ -8 & 9/2 \end{bmatrix}$$

$$C^{-1}A^{-1} = \begin{bmatrix} -7 & 4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 20 & -9 \\ \frac{29}{2} & -\frac{13}{2} \end{bmatrix}$$

$$\det(AC) = -13(40) - 18(-29) = -520 + 522$$
$$= 2.$$

$$(AC)^{-1} = \frac{1}{2}\begin{bmatrix} 40 & -18 \\ 29 & -13 \end{bmatrix} = \begin{bmatrix} 20 & -9 \\ \frac{29}{2} & -\frac{13}{2} \end{bmatrix}$$

$$(CA)^{-1} = A^{-1}C^{-1} = \begin{bmatrix} 9 & -5 \\ -8 & 9/2 \end{bmatrix}$$

Q2/ Put
$$M = \begin{bmatrix} 5 & 5 \\ 5 & -6 \end{bmatrix}.$$

Over $\mathbb{R}$,
$$\det M = \boxed{-30-25 = -55}$$

and
$$M^{-1} = \frac{1}{-55}\begin{bmatrix} -6 & -5 \\ -5 & 5 \end{bmatrix} = \frac{1}{55}\begin{bmatrix} 6 & 5 \\ 5 & -5 \end{bmatrix}$$

Over $\mathbb{Z}_7$,
$$M = \begin{bmatrix} 5 & 5 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix},$$

$$\det M = \boxed{-2-4 = -6 = 1}$$

and
$$M^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

Why does M^{-1} not exist over $\mathbb{Z}_{11}$?

$$\boxed{\det M = -55 = 0 \text{ in } \mathbb{Z}_{11}}$$

Q3/ Solve

$$\begin{aligned} x + \quad\;\; z &= 0 \\ y + z &= 0 \\ x + y \quad\;\; &= 1 \end{aligned}$$

over $\mathbb{R}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$$

with solution $(x, y, z) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$.

over $\mathbb{Z}_3$ by converting previous solution:

$$\boxed{(x, y, z) = (2, 2, 1)}$$

Why is there no solution over $\mathbb{Z}_2$?

$$\boxed{\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \text{inconsistent}}$$

Q4/ Solve the homogeneous system over $\mathbb{Z}_2$:

$$\begin{aligned} x_1 + x_2 + x_3 + x_5 &= 0 \\ x_1 + x_2 + x_4 + x_5 &= 0 \\ x_1 + x_3 + x_5 &= 0 \\ x_1 + x_2 + x_4 &= 0 \end{aligned}$$

after row reducing the following matrix to reduced row echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

with solution set

$$\{ (t, 0, t, t, 0) \mid t \in \mathbb{Z}_2 \}$$

$$= \{ (0,0,0,0,0), (1,0,1,1,0) \}.$$

Q5/ Consider the following permutations of

$$X = \{1, 2, 3, 4, 5\} :$$

$\alpha : 1 \mapsto 2, \; 2 \mapsto 5, \; 3 \mapsto 4, \; 4 \mapsto 1, \; 5 \mapsto 3$

$\beta : 1 \mapsto 3, \; 2 \mapsto 4, \; 3 \mapsto 1, \; 4 \mapsto 5, \; 5 \mapsto 2$

$\gamma : 1 \mapsto 5, \; 2 \mapsto 2, \; 3 \mapsto 4, \; 4 \mapsto 1, \; 5 \mapsto 3$

Express the following using cycle notation:

$\alpha = (1\ 2\ 5\ 3\ 4)$

$\beta = (1\ 3)(2\ 4\ 5)$

$\gamma = (1\ 5\ 3\ 4)(2) = (1\ 5\ 3\ 4)$

Simplify, using cycle notation:

$\alpha\beta = (1\ 2\ 5\ 3\ 4)(1\ 3)(2\ 4\ 5) = (1\ 4\ 3\ 5)$

$\beta\alpha = (1\ 3)(2\ 4\ 5)(1\ 2\ 5\ 3\ 4) = (1\ 4\ 3\ 2)$

$\beta\gamma = (1\ 3)(2\ 4\ 5)(1\ 5\ 3\ 4) = (1\ 4\ 3\ 5\ 2)$

$\gamma\alpha = (1\ 5\ 3\ 4)(1\ 2\ 5\ 3\ 4) = (1\ 3)(2\ 5\ 4)$

$\beta\gamma\alpha = (1\ 4\ 3\ 5\ 2)(1\ 2\ 5\ 3\ 4) = (1)(2)(3)(4)(5) = 1$

Q6/ Put $\alpha = (12)(34)$, $\beta = (13)(24)$,

$\gamma = \alpha\beta = (12)(34)(13)(24) = \boxed{(14)(23)}$

Find $\alpha^2 = \beta^2 = \gamma^2 = 1$

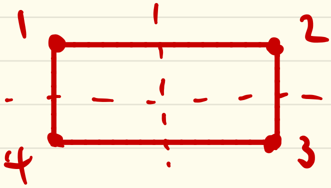
$\beta\gamma = (13)(24)(14)(23) = (12)(34)$

$\gamma\alpha = (14)(23)(12)(34) = (13)(24)$

and complete the multiplication table:

	1	α	β	γ
1	1	α	β	γ
α	α	1	γ	β
β	β	γ	1	α
γ	γ	β	α	1

How does this compare with the composition table for symmetries of the rectangle?



same if making identifications

α = vertical reflection

β = 180° rotation

γ = horizontal reflection

The group $G = \{1, \alpha, \beta, \gamma\}$ is abelian? T F

Q7/ Consider polynomials with coefficients from $\mathbb{Z}_2 = \{0,1\}$. Put

$$F = \{0, 1, x, 1+x\}.$$

Complete the addition table for F:

$+$	0	1	x	$1+x$
0	0	1	x	$1+x$
1	1	0	$1+x$	x
x	x	$1+x$	0	1
$1+x$	$1+x$	x	1	0

Complete the multiplication table for F if

(a) $x^2+1=0$:

$\cdot$	0	1	x	$1+x$
0	0	0	0	0
1	0	1	x	$1+x$
x	0	x	1	$1+x$
$1+x$	0	$1+x$	$1+x$	0

(b) $x^2+x+1=0$:

$\cdot$	0	1	x	$1+x$
0	0	0	0	0
1	0	1	x	$1+x$
x	0	x	$1+x$	1
$1+x$	0	$1+x$	1	x

Which of (a) or (b) yields a field? $\boxed{(b)}$