

MATH2022 Week 04
Worksheet Solutions

MATH 2022 Week 4 Worksheet

Q1/ Working over $\mathbb{Z}_3 = \{0, 1, 2\}$, put

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Find

$$\begin{aligned} \det M &= \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 2(2) - 0 = 1. \end{aligned}$$

Complete the following row reduction to find M^{-1} :

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 & 1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 \end{array} \right]$$

$$M^{-1} = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

Q2/ Working over $\mathbb{Z}_2 = \{0, 1\}$, put

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Find M^{-1} :

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array} \right], \text{ so } M^{-1} = \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{array} \right]$$

What is $\det M$?

$$\boxed{1}$$

(no working required)

Q3/ Working over $\mathbb{Z}_3 = \{0, 1, 2\}$,
 identify the e.r.o. in each case, and then
 find the corresponding elementary matrix:

$$(a) \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

e.r.o.

$$R_2 \rightarrow R_2 + R_1$$

elementary matrix $E_1 =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

e.r.o.

$$R_3 \rightarrow R_3 + 2R_1$$

elementary matrix $E_2 =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

e.r.o.

$$R_3 \rightarrow R_3 + R_2$$

elementary matrix $E_3 =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$(d) \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

elementary matrix $E_4 =$

e.r.o.

$$R_2 \rightarrow 2R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(e) \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

elementary matrix $E_5 =$

e.r.o.

$$R_3 \rightarrow 2R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$(f) \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

elementary matrix $E_6 =$

e.r.o.

$$R_1 \rightarrow R_1 + 2R_2$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(g) \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

elementary matrix $E_7 =$

e.r.o.

$$R_1 \rightarrow R_1 + R_3$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q4/ Working over $\mathbb{Z}_3$, again put

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Consider the elementary matrices from Q3:

$$E_1, E_2, E_3, E_4, E_5, E_6, E_7$$

Which of these are

lower triangular?

$$E_1, E_2, E_3, E_4, E_5$$

diagonal?

$$E_4, E_5$$

upper triangular?

$$E_4, E_5, E_6, E_7$$

Express I in terms of M and $E_1, \dots, E_7$:

$$I = E_7 E_6 E_5 E_4 E_3 E_2 E_1 M$$

Now express M in terms of $E_1^{-1}, \dots, E_7^{-1}$:

$$M = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1}$$

Find

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, E_6^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_7^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} E_1^{-1} E_2^{-1} E_3^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \end{aligned}$$

$$E_4^{-1} E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$E_6^{-1} E_7^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Write down an LDU-decomposition of M :

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q5/ True or false :

(a) $(2\ 3\ 4) = (2\ 3)(3\ 4)$

T **F**

(b) $(2\ 3\ 4) = (4\ 2)(4\ 3)$

T F

(c) $(1\ 2)(2\ 4) = (1\ 4)(2\ 1)$

T F

(d) $(2\ 3\ 4)$ is even

T F

(e) $(1\ 2)(3\ 4)(2\ 3) = (2\ 4\ 3\ 1)$

T **F**

(d) $(2\ 3\ 1\ 6\ 5)(4\ 2) = (6\ 5\ 1)(2\ 3\ 1\ 4)$

T F

(e) $(2\ 3\ 1\ 6\ 5)$ is even

T F

(f) $(3\ 6\ 5)(2\ 4\ 7)$ is odd

T **F**

(g) $(1\ 2)(3\ 4)(5\ 6)$ is even

T **F**

(h) $(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$ is odd

T **F**

Q6/ The starting configuration of the 8-puzzle is

1	2	3
4	5	6
7	8	

True or false?

1	2	3
4	6	8
7	5	

is possible

T F

2	1	3
4	5	6
7	8	

is possible

T **F**

2	1	3
6	4	5
8	7	

is impossible

T **F**

2	1	4
7		3
8	5	6

is impossible

T F