MATH2022 Week 06 Worksheet Solutions

MATH 2022 Week 6 Worksheet

Q1/ Put
$$M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

(a) Find

$$M \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$M \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) By inspection, from the previous part, what are the eigenvalues? $\boxed{1, 3}$

(c) Find P, P^{-1} and diagonal D such that $M = PDP^{-1}$.

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

(d) Find the characteristic polynomial

$$\chi(\lambda) = \det(\lambda I - M) = \begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix}$$

$$= (\lambda-2)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$$

(e) What are the roots of $\chi(\lambda)$?

$$\lambda = \boxed{1,\ 3}$$

(f) Find a general formula for M^n:

$$M^n = (PDP^{-1})^n = PD^nP^{-1}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -1 & 3^n \\ 1 & 3^n \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1+3^n & -1+3^n \\ -1+3^n & 1+3^n \end{bmatrix}$$

(g) Thus find $M^4 = \frac{1}{2}\begin{bmatrix} 82 & 80 \\ 80 & 82 \end{bmatrix} = \begin{bmatrix} 41 & 40 \\ 40 & 41 \end{bmatrix}$

Q2/ Put

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

(a) Find

$$M\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$M\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$M\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(b) By inspection, what are the eigenvalues?

$$\boxed{0, 1, 3}$$

(c) Find P and diagonal D such that $M = PDP^{-1}$:

$$P = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(d) Find P^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/3 & -1/3 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1/3 & 1/3 \end{array}\right]$$

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -1 & -1 \\ -3 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

(e) Find a general formula for M^n:

$$M^n = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^n \end{bmatrix}\frac{1}{3}\begin{bmatrix} 3 & -1 & -1 \\ -3 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0 & 0 & 3^n \\ 0 & -1 & 2(3^n) \\ 0 & 1 & 3^n \end{bmatrix}\begin{bmatrix} 3 & -1 & -1 \\ -3 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0 & 3^n & 3^n \\ 3 & 2(3^n) & -3+2(3^n) \\ -3 & 3^n & 3+3^n \end{bmatrix} = \begin{bmatrix} 0 & 3^{n-1} & 3^{n-1} \\ 1 & 2(3^{n-1}) & -1+2(3^{n-1}) \\ -1 & 3^{n-1} & 1+3^{n-1} \end{bmatrix}$$

(f) Thus find $M^4 = \begin{bmatrix} 0 & 27 & 27 \\ 1 & 54 & 53 \\ -1 & 27 & 28 \end{bmatrix}$

Q3/ Put $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(a) Find surd expressions for the eigenvalues :

$$\det(\lambda I - M) = \begin{vmatrix} \lambda-1 & -2 \\ -3 & \lambda-4 \end{vmatrix} = (\lambda-1)(\lambda-4)-6$$

$$= \lambda^2 - 5\lambda - 2 = 0 \quad \text{when}$$

$$\lambda = \frac{5 \pm \sqrt{25+8}}{2} = \frac{5 \pm \sqrt{33}}{2}.$$

(b) What is the dominant eigenvalue to 3 decimal places?

$$\lambda_1 = \boxed{5.372}$$

(c) What is the smaller eigenvalue to 3 decimal places?

$$\lambda_2 = \boxed{-0.372}$$

(d) Find

$$M^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

(e) Put $\underset{\sim}{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underset{\sim}{v}_{k+1} = M \underset{\sim}{v}_k$ $(k \geq 0)$

$$r_k = \frac{\text{1st entry of } \underset{\sim}{v}_k}{\text{1st entry of } \underset{\sim}{v}_{k-1}}$$

$$s_k = \frac{\text{2nd entry of } \underset{\sim}{v}_k}{\text{1st entry of } \underset{\sim}{v}_k}$$

Complete the following table to 3 d.p.

$\underset{\sim}{v}_0$	$\underset{\sim}{v}_1$	$\underset{\sim}{v}_2$	$\underset{\sim}{v}_3$	$\underset{\sim}{v}_4$	$\underset{\sim}{v}_5$	$\underset{\sim}{v}_6$	$\underset{\sim}{v}_7$
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 7 \\ 15 \end{bmatrix}$	$\begin{bmatrix} 37 \\ 81 \end{bmatrix}$	$\begin{bmatrix} 199 \\ 435 \end{bmatrix}$	$\begin{bmatrix} 1069 \\ 2337 \end{bmatrix}$	$\begin{bmatrix} 5743 \\ 12555 \end{bmatrix}$	$\begin{bmatrix} 30853 \\ 67449 \end{bmatrix}$
r_k	1	7	5.286	5.378	5.372	5.372	5.372
s_k	3	2.143	2.189	2.186	2.186	2.186	2.186

(f) Find $M \begin{bmatrix} 1 \\ s_7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2.186 \end{bmatrix} = \begin{bmatrix} 5.372 \\ 11.744 \end{bmatrix}$

(g) Read off the dominant eigenvalue $\approx \boxed{5.372}$

(h) Put $\underset{\sim}{w}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underset{\sim}{w}_{k+1} = M^{-1}\underset{\sim}{w}_k = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \underset{\sim}{w}_k$

$$t_k = \frac{\text{1st entry of } \underset{\sim}{w}_k}{\text{1st entry of } \underset{\sim}{w}_{k-1}}$$

$$u_k = \frac{\text{2nd entry of } \underset{\sim}{w}_k}{\text{1st entry of } \underset{\sim}{w}_k}$$

Complete the following table to 3 d.p.

$\underset{\sim}{w}_0$	$\underset{\sim}{w}_1$	$\underset{\sim}{w}_2$	$\underset{\sim}{w}_3$	$\underset{\sim}{w}_4$
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -2 \\ 1.5 \end{bmatrix}$	$\begin{bmatrix} 5.5 \\ -3.75 \end{bmatrix}$	$\begin{bmatrix} -14.75 \\ 10.125 \end{bmatrix}$	$\begin{bmatrix} 39.625 \\ -27.188 \end{bmatrix}$
t_k	-2	-2.75	-2.682	-2.686
u_k	-0.75	-0.682	-0.686	-0.686

(i) Find $M \begin{bmatrix} 1 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -0.686 \end{bmatrix} = \begin{bmatrix} -0.372 \\ 0.256 \end{bmatrix}$

(j) Read off the negative eigenvalue of $M \approx \boxed{-0.372}$

Q4/ Suppose that M is an invertible matrix with eigenvector $\underset{\sim}{v}$ corresponding to eigenvalue λ.

(a) Prove that $\lambda \neq 0$.

Proof: We have $M\underset{\sim}{v} = \lambda \underset{\sim}{v}$ and $\underset{\sim}{v} \neq \underset{\sim}{0}$.

If $\lambda = 0$ then $M\underset{\sim}{v} = 0\underset{\sim}{v} = \underset{\sim}{0}$,

so
$$\underset{\sim}{v} = I\underset{\sim}{v} = M^{-1}M\underset{\sim}{v} = M^{-1}\underset{\sim}{0} = \underset{\sim}{0},$$

contradicting that $\underset{\sim}{v} \neq \underset{\sim}{0}$. Hence $\lambda \neq 0$.

(b) Prove that $\underset{\sim}{v}$ is an eigenvector for M^{-1} corresponding to eigenvalue λ^{-1}.

Proof: We have

$$M^{-1}\underset{\sim}{v} = M^{-1}(\lambda^{-1}\lambda\underset{\sim}{v}) = \lambda^{-1}M^{-1}(\lambda\underset{\sim}{v})$$

$$= \lambda^{-1}M^{-1}(M\underset{\sim}{v}) = \lambda^{-1}I\underset{\sim}{v} = \lambda^{-1}\underset{\sim}{v},$$

i.e. $M^{-1}\underset{\sim}{v} = \lambda^{-1}\underset{\sim}{v}$, as required.