MATH2022 Week 07
Worksheet

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Q1/ Solve the following system over $\mathbb{Z}_3$:

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \\
x_1 \qquad + x_3 \qquad + x_5 &= 1 \\
x_1 + 2x_2 + x_3 + 2x_4 + x_5 &= 2 \\
2x_2 \qquad + 2x_4 \qquad &= 1
\end{aligned}$$

What is the size of the solution set?

$\boxed{}$

Q2/ Put $M = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}$,

$A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Find elementary matrices E_1, E_2, E_3 such that

(a) $E_1 M = A$ $\quad$ $\boxed{E_1 = \qquad\qquad}$

(b) $E_2 A = B$ $\quad$ $\boxed{E_2 = \qquad\qquad}$

(c) $E_3 B = I$ $\quad$ $\boxed{E_3 = \qquad\qquad}$

Express M as a product of elementary matrices:

Q3/ Consider the following stochastic matrix:

$$M = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$$

(a) Row reduce $I - M$ and find the eigenspace corresponding to eigenvalue 1:

(b) Find the unique steady-state probability vector $\underset{\sim}{v}$:

Q4/ Consider the following stochastic matrix:

$$M = \begin{bmatrix} 0.3 & 0 & 0.2 \\ 0.3 & 0.5 & 0.3 \\ 0.4 & 0.5 & 0.5 \end{bmatrix}$$

Rows reduce $I - M$ and hence find the unique steady-state probability vector $\underset{\sim}{s}$:

$I - M =$

Q5/ Consider the following where $\theta \in \mathbb{R}$:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad T_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Simplify the following :

(a) $T_\theta T_\varphi = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}$

$=$

(b) $T_{\pi/2} T_{\pi/3} =$

(c) $R_{\pi/3} T_{\pi/2} R_{\pi/3} =$

(d) $T_{\pi/3} R_{\pi/2} T_{\pi/3} =$

(e) $R_{2\pi/3} T_{\pi/2} R_{\pi/4} T_{\pi/3} =$

Q6/ A finite group G is cyclic if all elements can be obtained by applying the group operation repeatedly to a single element g, called a generator, in which case we write

$$G = \langle g \rangle .$$

Find all z such that $\mathbb{Z}_n = \langle z \rangle$ with respect to addition in the following cases:

(a) $\mathbb{Z}_3 = \langle z \rangle$ where $z = \boxed{}$

(b) $\mathbb{Z}_4 = \langle z \rangle$ where $z = \boxed{}$

(c) $\mathbb{Z}_5 = \langle z \rangle$ where $z = \boxed{}$

(d) $\mathbb{Z}_6 = \langle z \rangle$ where $z = \boxed{}$

(e) $\mathbb{Z}_8 = \langle z \rangle$ where $z = \boxed{}$

Q7/ Let $G = \mathbb{Z}_7 \setminus \{0\} = \{1,2,3,4,5,6\}$.

Then G is a group under multiplication.

Find all successive distinct powers of z in each case (working over $\mathbb{Z}_7$):

$$z, \; z^2, \; z^3, \; z^4, \; \ldots$$

(a) $z = 1$: $\boxed{}$

(b) $z = 2$: $\boxed{}$

(c) $z = 3$: $\boxed{}$

(d) $z = 4$: $\boxed{}$

(e) $z = 5$: $\boxed{}$

(f) $z = 6$: $\boxed{}$

True or False: G is cyclic. T F

Q8/ Let $G = \{3, 6, 9, 12\}$.

Fill out the following multiplication table working over $\mathbb{Z}_{15}$:

$\cdot$	3	6	9	12
3				
6				
9				
12				

(a) True or False: G is a group. T F

(b) What is the identity element? $\boxed{}$

(c) Find $3^{-1} = \boxed{}$, $9^{-1} = \boxed{}$

(d) True or False: G is cyclic. T F

(e) Find all g such that $G = \langle g \rangle$:

$g = \boxed{}$